Module 7: Fair Division

This material corresponds to
chapter 13 of the textbook,
For All Practical Purposes

FAIR DIVISION
CHAPTER 13

TIME FRAME: 9 Days

ENDURING UNDERSTANDINGS:

Mathematics can help provide procedures to ensure fair and equitable resolutions of conflicts over property and rights.

ESSENTIAL (ASSESSMENT) QUESTIONS:

1. How can an estate be divided fairly among several inheritors?
2. How can a cake dividing procedure be proportional but not envy-free?

CRMS:

PS/Reasoning: 1.2
Communication: 2.3

AT THE END OF THE MODULE STUDENTS WILL KNOW AND BE ABLE TO:

1. Describe various cake dividing procedures.
2. Explain the Adjusted Winner procedure and its possible applications.
3. Analyze if a cake dividing procedure is envy free.

PRE-REQUISITE KNOWLEDGE/SKILLS: Fractions, Percents

PRE-ASSESSMENT: None

ACTIVITIES:

Have Your Cake and Eat it Too worksheet
Mom Promised Me the House! (Requires access to a computer lab.)
Equal Might Not Be Fair worksheet
Cutting Procedure Analysis worksheet
Candy Dividing activity

POST-ASSESSMENTS:

Team paper on problem #2 of Writing Projects (**7e**) p. 502 / (**8e**) p. 431 in FAPP.
Reflection paper on Fair Division that includes where one finds fair division problems (use the articles collected the first day for help) and how they might be solved using different methods.

LITERACY STRATEGIES INTRODUCED:

Newspaper and magazine article research

RESOURCES:

Project TIME DVD - *NUMB3RS* clip: "*One Hour*"
Project TIME CD-ROM spreadsheet: **Estate Division.xls**
Article on Pareto Optimality

DAILY PLAN

Day 1

- Show *NUMB3RS* clip *"One Hour"* from Project TIME DVD.
- Distribute worksheets: **Have Your Cake and Eat it Too**.
- Discuss Fairness.
- In-class have students do problems #1- 4 on *NUMB3RS* activity.

HW: Read Article on *"Biography of Vilfredo Pareto"*
Newspaper Assignment: Find an article in local or national paper that involves conflict over property or rights. In other words, find a Fair Division problem.
Read **7e** pp.475-480 / **8e** pp.407-413 taking Cornell Notes on those pages.

Day 2

- Share-One-Around protocol on answers to *NUMB3RS* activity.
- Concept Map review – have students work in groups and report out.
- Go over homework and post articles about Fair Division.
- Give direct instruction on the Adjusted Winner Allocation.

HW: **7e** p.497: #1-7 (any or all)
8e p.427: #1-7 (any or all)

Days 3-4

- Go over homework.
- Give direct instruction on Knaster Inheritance Procedure (do one by hand, discuss why the extra is divided three ways).
- Students work on spreadsheet activity: **Mom Promised Me the House!** (This activity requires access to a computer lab, since students will be using spreadsheet software.) If students are at a level where they might program spreadsheets themselves, you may have them write the spreadsheet application. Otherwise, find the prepared spreadsheet on the Project TIME CD-ROM.

HW: **7e** p.498: #8-12 (any or all)
8e p.429: #8-12 (any or all)

Day 5

- Go over homework.
- Cake Cutting activity: **Equal Might Not Be Fair**.
- Groups share out; students take Cornell Notes over ideas covered in presentations.
- Assign Groups Jigsaw activity. Two groups are assigned "**Lone Divider**", two groups the "**Last Diminisher**", and two groups the "**Selfridge-Conway Envy-Free Procedure for Three Players**".

HW: Each student fills out Analysis Sheet for his or her assigned method. Assign **7e** p.498: #13 -18 (any or all) / **8e** p.428: #13 -18 (any or all)

Days 6-7

- Go over homework from textbook.
- Allow students 10 minutes for groups to finalize presentations.
- Choose one group for each division method to present. The other group for each method answers questions from the class about the method. Students take Cornell Notes on the divider methods. Presentations should take 10-12 minutes including question time.

HW: **7e** pp.499-502 / **8e** pp.429-431. Choose two problems from each of the categories Divide and Choose, Proportionality, and The Problem of Envy.

Day 8

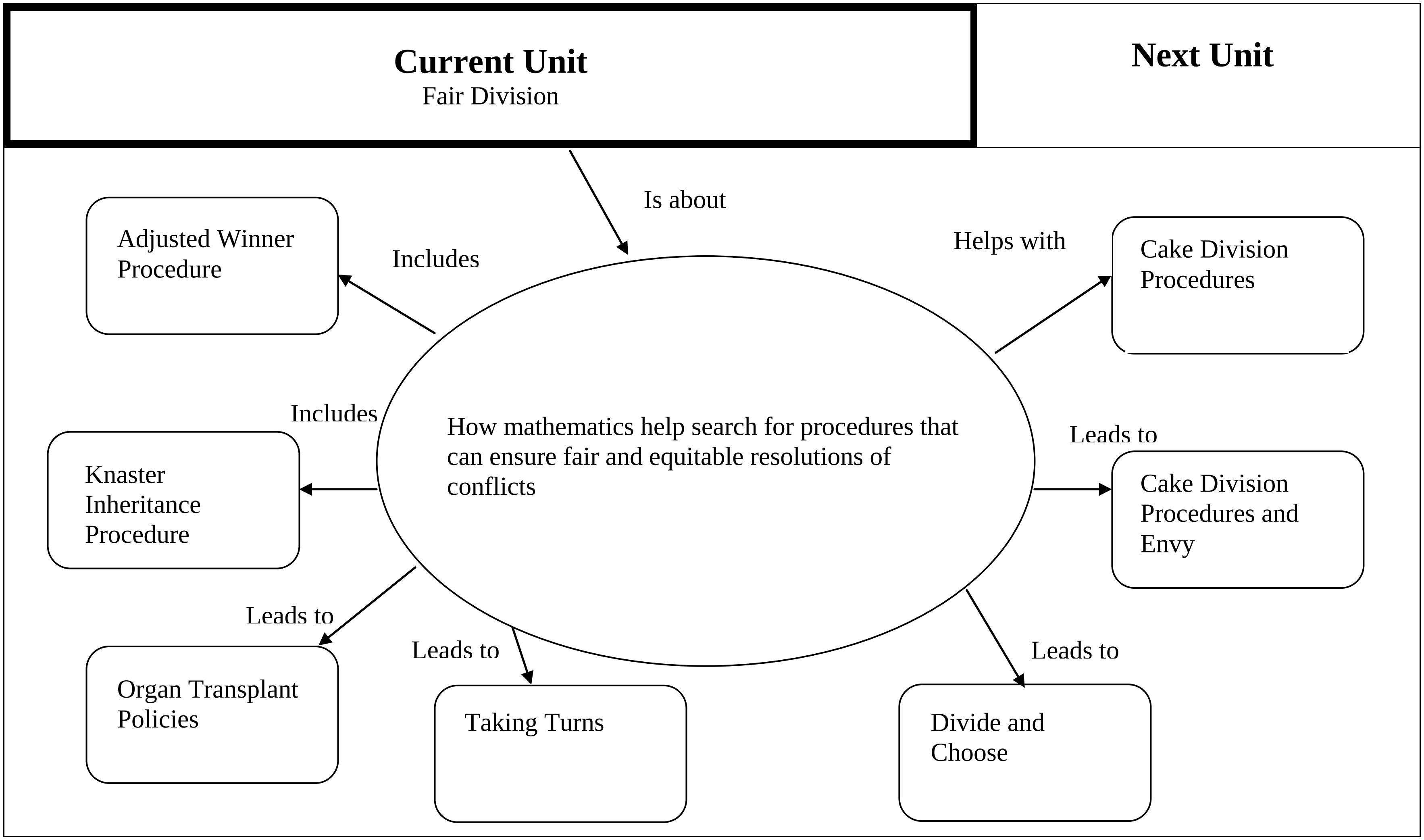
- Go over homework.
- **Candy Dividing** activity.
- **Organ Transplant 7e –** You may copy the **8e** pp. 413-415: #13-14
- Assign pairs from groups to work on writing assignment #2 on **7e** p. 502 / **8e** p. 431 from FAPP.

HW: Students prepare team papers on #2. Individuals work on reflection paper.

Day 9

- Reflection Paper on Fair Division: This should be a one or two page paper. A suggestion for content is as follows:
  Paragraph 1: A summary of what Fair Division means.
  Paragraph 2: In the real world, where do we find Fair Division problems?
  Paragraph 3: Pick a particular Fair Division problem from the articles submitted to the class and discuss the problem.
  Paragraph 4: How might the chosen problem be solved using the methods studied in class?
- Have students finish and turn in the writing project as the final assessment.

Concept Map



Current Unit
Fair Division
Next Unit
Is about
How mathematics help search for procedures that can ensure fair and equitable resolutions of conflicts
Includes
Adjusted Winner Procedure
Includes
Knaster Inheritance Procedure
Leads to
Organ Transplant Policies
Leads to
Taking Turns
Leads to
Divide and Choose
Leads to
Cake Division Procedures and Envy
Helps with
Cake Division Procedures

NUMB3RS ACTIVITY: HAVE YOUR CAKE AND EAT IT TOO

Description and Answer Key

The worksheet on the following page, and the sample answers below, are taken from:

http://education.ti.com/go/NUMB3RS

The instructor should play the video clip "*One Hour*" from the Project TIME DVD and then distribute copies of the worksheet to the class. Ask students to work in small groups.

The purpose of this activity is to introduce and motivate the concept of "fair division" procedures than can be used when different parties trying to apportion a good or goods value them differently from one another.

SAMPLE SOLUTIONS TO WORKSHEET QUESTIONS:

1a. Cut on the far left side; the right side is the entire cake.

1b. Cut on the far right side; the left side is the entire cake.

1c. As Charlie slides the knife over the cake from left to right, his value on the left side increases from 0% to 100%. Because the cake is one continuous piece with an infinite number of possible cuts, there must be some cut that values the left side at 50%. The right side will also be 50% for this cut.

2. Charlie has already stated that the two pieces are of equal value, so he will be happy with either piece. If Don considers the pieces of equal value, he can take either and will be happy. If he feels one is worth more than the other, he can take that one and Charlie still gets a piece he considers equal in value to Don's. Notice that Don has actually received what he considers to be more than a fair share.

3a. This method is not fair. Amita chooses last, so she has incentive to cut equal pieces. Charlie will pick the piece that he favors the most, so Don may be forced to choose between two pieces that he does not consider to be 1/3 of the original.

3b. This method is fair, though cumbersome and potentially messy and less satisfying than an intact piece of cake.

3c. This method is fair and can be extended to any number of people.

4. Answers will vary.

Name ____ Date _____________

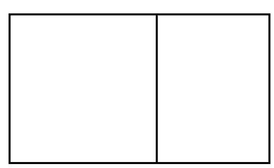
NUMB3RS Activity: Have Your Cake and Eat It Too

In “One Hour,” the FBI is chasing kidnappers who demand a ransom of $3.2 million. The FBI thinks that this number has a personal significance to the kidnapper, but Charlie disagrees. He compares the division of the money among the crew to the distribution of a cake among a group. Charlie explains that if two people agree to divide the cake, “the obvious strategy is to cut it down the middle. But if one person likes frosting more than the other, one person prefers vanilla cake or one person prefers chocolate cake with vanilla frosting, it may be that slices unequal in size actually create a fairer distribution.” Charlie uses cake-cutting algorithms to conclude that the head kidnapper’s goal is to get $1.65 million, an amount with a very personal meaning to one person involved. In this activity, you will explore fair ways for two people to share a cake and then extend the exploration for three people.

Consider the case of two people who agree to fairly divide a cake. A fair share means that both people consider the piece they receive to be at least half the value of the cake. As Charlie mentions in the episode, people can have different opinions about the value of a piece. For example, someone who likes frosting may put a high value on a piece of cake with an abundance of frosting, even if it has less volume.

One person makes a cut to divide the cake into what he or she considers to be two equal parts; the other person chooses the first piece. This is called the “I cut you choose” method. The first person is likely to cut the pieces as equally as possible, because he or she could otherwise be stuck with a less desirable piece.

Charlie and Don are using the “I cut, you choose” method to divide a cake. Charlie will make one vertical cut. Each cut will create two pieces, one on the left and one on the right. Charlie will consider each piece to be worth a percentage of the whole. For example, for the cut shown below, he may feel that the left side is worth 40% of the cake and the right side is worth 60% of the cake. (Remember that Charlie is making a judgment based on his personal preference, not necessarily on dimensions.)



1. **a.** Where has Charlie placed the cut if he considers the left piece to be 0%?

b. Where has Charlie placed the cut if he considers the left piece to be 100%?

c. Explain why there must be one cut that Charlie can make so that he values the two pieces equally.

The answer to Question 1c involves the Intermediate Value Theorem, a concept explored in calculus. This theorem states that for a function f that is continuous on the interval $[a, b]$, if there exists a value d between $f(a)$ and $f(b)$, then there is a value of c in (a, b) such that $f(c) = d$. For example, if someone was 5 feet tall last year and is now 5 feet 2 inches tall, at some point that person was 5 feet 1 inch tall. With the same reasoning, there is at least one place a person can cut a cake to create two pieces of equal value. The trick, of course, is to find that place. The

Kathy Erickson, Monument Mountain Regional High School, Great Barrington, MA

Intermediate Value Theorem is called an existence theorem, because it guarantees that a value exists, but does not give a method to find that value.

In the two-person scenario, the first person determines where to cut the cake. Charlie cuts the cake into what he considers two pieces of equal value. Don will choose either piece, and Charlie will receive the other.

2. Explain why the brothers both receive a piece they believe to be at least 50% of the cake.

How does the scenario change if three people are trying to fairly divide a cake? A fair division in this case means each person believes that he or she receives at least a third of the value of the cake.

3. Decide if each method below is fair.

a. Amita cuts the cake into what she considers to be equal thirds. Charlie chooses the first piece, Don chooses second piece, and Amita receives the remaining piece.

b. Charlie cuts the cake into what he considers equal halves. Amita chooses one piece and Charlie keeps the other. Amita and Charlie each cut their piece into what they consider to be equal thirds. Don chooses one piece from Charlie and one piece from Amita. Charlie and Amita keep what is left.

c. Amita indicates a piece she considers to be worth a third of the value of the cake. Charlie and Don have the opportunity to inspect the piece. If either one thinks the piece represents more than a third, he indicates how to trim the piece so the value, in his opinion, is only one-third of the whole. The portion is given to the person whose cut is accepted by the other two people. After the first piece is taken, the remaining two people use a similar process on what remains of the cake.

4. Develop your own method for fairly dividing a cake between three people.

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Biography of Vilfredo Pareto (1848-1923)

Pareto is best known for two concepts that are named after him. The first and most familiar is the concept of Pareto optimality. A Pareto-optimal allocation of resources is achieved when it is not possible to make anyone better off without making someone else worse off. The second is Pareto's law of income distribution. This law, which Pareto derived from British data on income, showed a linear relationship between each income level and the number of people who received more than that income. Pareto found similar results for Prussia, Saxony, Paris, and some Italian cities. Although Pareto thought his law should be "provisionally accepted as universal," he thought that exceptions were possible, and as it turns out, many exceptions have been found.

Pareto is also known for showing that the assumption that the utility of goods can actually be measured was not necessary for deriving any of the standard results in consumer theory. He showed that by simply being able to rank bundles of goods, consumers would act as economists had said they would.

In his later years Pareto shifted from economics to sociology. This reflected his own change in beliefs about how humans act. He came to believe that men act nonlogically, "but they make believe they are acting logically."

Born in Paris to Italian exiles, Pareto moved to Italy to complete his education in mathematics and literature. After graduating from the Polytechnic Institute in Turin in 1869, he applied his prodigious mathematical abilities as an engineer for the railroads. Throughout his life Pareto was an active critic of the Italian government's economic policies. He published pamphlets and articles denouncing protectionism and militarism, which he viewed as being the two greatest enemies of liberty. Although he was keenly informed on economic policy and frequently debated it, Pareto did not study economics seriously until he was forty-two. In 1893 he succeeded his mentor, Walras, as chair of economics at the University of Lausanne. His principal publications are *Cours d'économie politique* (1896-97), Pareto's first book, which he wrote at age forty-nine, and *Manual of Political Economy* (1906).

A self-described pacifist who disdained honors, Pareto was nominated in 1923 to a Senate seat in Mussolini's fledgling government but refused to become a ratified member. He died that year and was buried without fanfare in a small cemetery in Celigny.

Selected Works

"The New Theories of Economics." *Journal of Political Economy* 5: pp.485-502.

Trattato di Sociologia generale. 1916.

The Mind and Society. 1935. Four volumes. Translated by Andrew Bongiorno and Arthur Livingston, with the advice and active cooperation of James Harvey Rogers. Includes introductory note with references to 1915 note on Pareto (*Nation*) and subsequent references by Joan Robinson, Will Durant, Aldous Huxley, and more.

Vol. I. Non-Logical Conduct
Vol. II. Analysis of Sentiment (Theory of Residues)
Vol. III. Sentiment in Thinking (Theory of Derivations)
Vol. IV. The General Form of Society

From http://www.econlib.org/library/Enc/bios/Pareto.html

MOM PROMISED ME THE HOUSE!

Instructions and Answer Key

After the teacher describes the Inheritance Procedure, she should distribute copies of the worksheet on the following pages. Then students should explore the spreadsheet "*Estate Division*" that is included on the Project TIME CD-ROM and try to use it to answer the questions on the worksheet.

Students have a hard time with splitting the money up at the very end of the problem. After Bonita pays into the estate the extra money from the value of the house ($400,000 – value of her fair share) and after the other two children are compensated from that pot so they have their fair share, there is always money left over. Students think the money should go back to Bonita since she put it in, but instead it gets divided three ways. Once Bonita pays the estate it is no longer her money, but the estate money and all the money gets divided equally. If not, Abby and Carlos would have gotten exactly their view of their fair share, but Bonita would have gotten far more than her view of her fair share.

SOME SAMPLE ANSWERS TO WORKSHEET QUESTIONS

1. What was the value of the estate in each person's eyes? What was their fair share from their own perspective?

To Abby, the total estate was worth $434,500, so her fair share was $144,833.33. To Bonita, it was worth $434,600, so her fair share was $144,866.67. To Carl, it was worth $426,000, so his fair share was $142,000.

2. Who received each object?

Bonita gets the house and the boat. Abby gets the car. Carlos doesn't get any item, just money. How would the money be divided? *The $30,000 would be divided three ways. Usually the money isn't divided until the end when everyone has paid their share into the estate.*

3. After everyone receives his/her property and after each has paid into the estate any money owed, what does each child end up with? Do the values of these objects and money equal more than what each believed was a fair share? Explain.

Answers will vary.

4. What should happen if two people bid the same price on an object?

A random drawing of names is one way to decide who gets the property.

5. Suppose that Carlos really doesn't want the car (too expensive for upkeep, and it isn't running very well anyway) and decides to bid $0 for the car? How does this hurt him in the long run?

His fair share would go down. He would still get any items where he had the largest bid, but his share of the money would be less. Why should a person not bid $0 on an object? If a person bids 0 on an item it makes the value of the estate in his eyes lower, therefore his fair share will be lower.

6. Suppose two of the children (Abby and Carlos) go into collusion with each other. They do this by making their bids on each object, but before they turn them in, they confer between themselves. They honor each others' winning bid, but raise their own losing bids up to just under the winning bid. In other words, Carlos should get the house, so Abby respects Carlos' bid but raises her bid to $372,000. Abby gets the car so her bid stays the same, but Carlos raises his bid to $51,000. As far as the boat is concerned Abby gets the boat, but again Carlos raises his boat bid to $2400. In this scheme everyone still gets the objects they would have in the original problem, but what happens to their final settlement. Why does this happen?

By raising the losing bids the value of the entire estate goes up and therefore one's fair share goes up.

Here is a screenshot of the spreadsheet:

Estate Division.xls [Compatibility Mode]

	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1	**Abby**	**Bonita**	**Carlos**		**Max Bids**								**House Awards**				
2	350000	355000	345000		355000		Calculations in columns N--R compute						A	B	C		
3	16000	14000	17000		17000		awards in case of equal bids.						0	2	0		2
4	8000	5000	8000		8000								0	355000	0		
5	6000	6000	6000				Note: When using this spreadsheet, only edit						**Car Awards**				
6							cells in columns B, C and D through row 4.						A	B	C		
7	380000	380000	376000				Cell A5 is also editable.						0	0	4		4
8	126666.67	126666.67	125333.33										0	0	17000		
9													**Boat Awards**				
10	0	355000	0										A	B	C		
11	0	0	17000										1	0	4		5
12	8000	0	0										8000	0	0		
13	2000	2000	2000														
14																	
15	10000	357000	19000														
16	126666.67	126666.67	125333.33														
17	-116666.7	230333.33	-106333.3		Kitty Value=												
18					7333.3333												
19	2444.4444	2444.4444	2444.4444														
20																	
21	**Abby**	**Bonita**	**Carlos**														
22	**129111.11**	**129111.11**	**127777.78**											Random Variables:			
23															2-way tie	3-way tie	
24														House	0.21564	0.3303	
25															1	2	
26																	
27																	
28														Car	0.85571	0.1662	
29															2	2	

Sheet1 Sheet2 Sheet3

MOM PROMISED ME THE HOUSE!

Purpose: To show one way (Knaster Inheritance Procedure) to divide up an inheritance among three or more people if the objects in the inheritance are discrete objects (house, car, boat) as opposed to continuous objects (money, land).

Three children Abby, Bonita, and Carlos must divide up the inheritance from their parents. In the estate are the family home, a new sports car, and an old fishing boat with outboard motor and fishing gear. There is also $30,000 in cash.

Since none of the children can agree on how it should be divided and they do not want to sell everything and divide it three ways, they decide to follow the Knaster Inheritance or Sealed Bid procedure. In secret they write their bids (what they value the objects at) on sealed papers. After all bids are turned in they are opened and whoever valued each object the highest will get that object.

Below are the bids for each person:

Child	**House**	**Car**	**Boat**	**Money**
Abby	350,000	52,000	2500	30,000
Bonita	400,000	43,000	3000	30,000
Carlos	375,000	20,000	1000	30,000

Enter these values into the Spreadsheet Application. From the spreadsheet answer the following questions.

1. What was the value of the estate in each person's eyes? What was their fair share from their own perspective?

2. Who received each object? How would the money be divided?

3. After everyone receives his/her property and after each has paid into the estate any money owed, what does each child end up with? Do the values of these objects and money equal more than what each believed was a fair share? Explain.

4. What should happen if two people bid the same price on an object?

5. Suppose that Carlos really doesn't want the car (too expensive for upkeep, and it isn't running very well anyway) and decides to bid $0 for the car? How does this hurt him in the long run? Why should a person not bid $0 on an object?

6. Suppose two of the children (Abby and Carlos) go into collusion with each other. They do this by making their bids on each object, but before they turn them in, they confer between themselves. They honor each other's winning bid, but raise their own losing bids up to just under the winning bid. In other words, Carlos should get the house, so Abby respects Carlos' bid but raises her bid to $372,000. Abby gets the car so her bid stays the same, but Carlos raises his bid to $51,000. As far as the boat is concerned Abby gets the boat, but again Carlos raises his boat bid to $2400. In this scheme everyone still gets the objects they would have in the original problem, but what happens to their final settlement? Why does this happen?

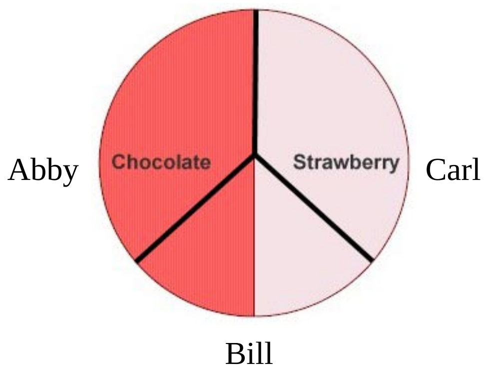
FAIR DIVISION: EQUAL MIGHT NOT BE FAIR

Description and Solutions

PURPOSE: This activity reviews the use of fractions, angle measurements, and simple equations. It also shows how a cake can be divided equally in size, but the value of each piece depends on the likes and dislikes of each of the people involved.

In a 3-person team, students will take a sample cake division and decide how much their own share was worth. Two teams will have one cake division; two other teams will have the second cake division, and so on. The teams with the same cake will share their results with each other. Share out strategies and/or ideas in class report-out procedure.

Cakes:



Have students divide into groups of three and each person takes on the role of *A*, *B*, or *C*. Each student should figure the value of their share, does it equal $4 dollars, and how could he/she convince the others to help him get a slice that is valued at $4 at a minimum.

Have teams with the same cake share with each other. Then have all teams share out discoveries (while someone takes class notes and individuals take Cornell Notes). After the share out, review major points and also review mathematics involved in cake division. Homework is review sheet of computation with fractions, decimals, percents, ratios, and angles.

Points to look for when people share out strategies (Not solutions since there might be so many of them and because groups have different problems. Solutions are only interesting to groups with the same problems).

- Solutions are not unique.
- If a person values one type of frosting as 3 times the other, then work with equation $3x + 1x = \$12$. The value is in a ratio of 3 to 1 so there are 4 parts total. Students may also guess and check to get the values of each side of the cake.
- Extension might be to make Abby, Bill, and Carl's piece worth exactly the same.

SOLUTIONS TO VERSION A

1. What is the central angle of Bill's piece? Of the strawberry part?

120° and 60°

2. How much is the strawberry half and the chocolate half valued by each person? Justify your answer.

$9 for strawberry and $3 for chocolate from Abby's point of view. From Bill both sides are the same at $6 and from Carl's point of view, the chocolate half is worth $8 and the strawberry is worth $4.

3. What is the dollar value of Abby's piece to Abby, Bill's piece to Bill, and Carl's piece to Carl? Justify your answer.

Abby values her share at 2/3 of $3 or $2. Bill values his share at $4 and Carl values his share at 2/3 of $4 or $2.67.

4. Is everyone happy with his/her share? How could one or both of the other two cut their share and/or trade parts with the unhappy person so that each person has a $4 share from his/her perspective?

Abby is unhappy and so is Carl. Have Abby trade with Carl.

5. Is there a way to divide the cake so that each person has a share that they value over $4? How?

Abby could give Bill a 10° slice (worth .50 to her) so she would have a $5.50 share in her eyes, Carl would have $5.33 share, and in Bill's eyes he would have a (each degree of cake is worth 1200/360 cents per degree or 3 1/3 cents) $4.33 share.

SOLUTIONS TO VERSION B

1. What is the central angle of Bill's piece? Of the strawberry part?

120° and 60°

2. How much is the strawberry half and the chocolate half valued by each person?

To Abby the chocolate half is worth $8 and the strawberry is worth $4. To Bill each have is worth the same. To Carl the chocolate is worth $9 and the strawberry is worth $3.

3. What is the dollar value of Abby's piece to Abby, Bill's piece to Bill, and Carl's piece to Carl? Justify your answer.

Abby's value to her is 2/3 of $8 or $5.33 and Bill's piece is worth 2/3 of $6 or$4, and to Carl his share is worth 2/3(3) or $2.

4. Is there an unhappy person? How could one or both of the other two cut their share and/or trade parts with the unhappy person so that each person has at least $4 share from his/her perspective?

Everyone is happy and each person is receiving at least $4.

5. Is there a way to divide the cake so that each person has a share that they value over $4? How?

Start by dividing the chocolate part in half, Carl gets half valued at $4.50 and Abby gets half valued at $4. Give Bill 2/3 of strawberry half and he has value of $4. Now take the ¼ of cake left (strawberry half) and divide it (however you want) between Abby and Bill so they each now have a portion valued at more than $4.

SOLUTIONS TO VERSION C

1. What is the central angle of Bill's piece? Of the strawberry part?

120° and 60°

2. How much is the strawberry half and the chocolate half valued by each person?

To Abby the chocolate half is worth $12 and the strawberry half is $0. To Bill both strawberry and chocolate are worth $6 each. To Carl the chocolate is worth $9 and the strawberry is worth $3.

3. What is the dollar value of Abby's piece to Abby, Bill's piece to Bill, and Carl's piece to Carl? Justify your answer.

*$8 or (2/3*12) to Abby, Bill values his piece at $4, and Carl values his piece at $2 or (2/3*3).*

4. Is there an unhappy person? How could one or both of the other two cut their share and/or trade parts with the unhappy person so that each person has at least a $4 share from his/her perspective?

Carl is really unhappy. Bill could give Abby his pieces, which Abby would value at $4, and she could give Carl her chocolate piece which he would value at $6. Bill would receive the strawberry piece at a value of $4.

5. Is there a way to divide the cake so that each person has a share that they value over $4? How?

Carl could give Abby 20° slice of his chocolate (which he values at $1). Carl's piece would be worth $5 and Abby would view the 20° slice as $1.33 (each degree of chocolate is worth 1200/180 or about 6 2/3 cents per degree. Abby's slice would then be worth $5.33. Abby could give Bill her strawberry piece which would give Bill a value of $6 total.

SOLUTIONS TO VERSION D

1. What is the central angle of Bill's piece? *120°* Of the strawberry part? *60°*

2. How much is the strawberry half and the chocolate half valued by each person?

Abby values the strawberry at $8 and the chocolate at $4. Bill values the strawberry as $9 and chocolate at $3. Carl views both sides equal at $6.

3. What is the dollar value of Abby's piece to Abby, Bill's piece to Bill, and Carl's piece to Carl? Justify your answer.

Abby values her piece as 2/3 of $4 or $2.67, Carl values his piece at 2/3 of $6 or $4, and Bill views his piece as 1/3 ($9) + 1/3 ($3) or $4

4. Is there an unhappy person? How could one or both of the other two cut their share and/or trade parts with the unhappy person so that each person has a $4 share from his/her perspective?

Abby is very unhappy. Abby and Carl could swap giving Carl a value of $4 and Abby a value of $5.33.

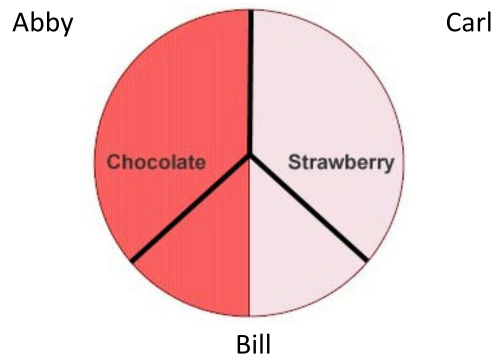
5. Is there a way to divide the cake so that each person has a share that they value over $4? How?

There are many possibilities here. Abby could take ½ the strawberry valued at $4 for her. Carl could have 2/3 of the chocolate valued at $4 for him. Bill could take 1/3 of the chocolate and 1/3 of the strawberry valued at $4 for him. There is 1/6 of the strawberry portion of the cake left. Divide this between the three people.

FAIR DIVISION: EQUAL MIGHT NOT BE FAIR

VERSION A

Purpose: This activity reviews the use of fractions, angle measurements, and simple equations. It also shows how a cake can be divided equally in size, but the value of each piece depends on the likes and dislikes of each of the people involved. Note: Chocolate is on the left-hand side; strawberry is on the right-hand side of the cake.



The cake above costs \$12 and is covered in half strawberry and half chocolate frosting. The central angle on each of Abby and Carl's slice is 120°. Abby likes strawberry three times as much as chocolate. Bill likes both frostings the same. Carl likes chocolate twice as much as strawberry. Since the cake cost \$12, each person's fair share should be worth \$4 in his or her eyes.

1. What is the central angle of Bill's piece? Of the strawberry part?

2. How much is the strawberry half and the chocolate half valued by each person?

3. What is the dollar value of Abby's piece to Abby, Bill's piece to Bill, and Carl's piece to Carl? Justify your answer.

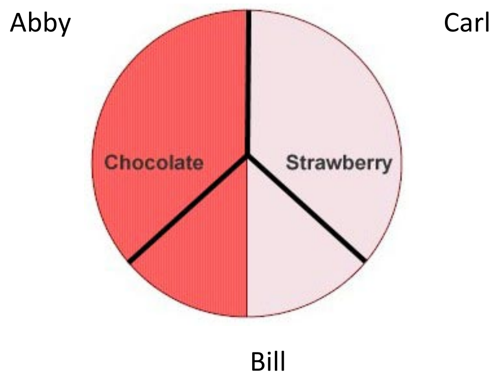
4. Is there an unhappy person? How could one or both of the other two cut their share and/or trade parts with the unhappy person so that each person has a \$4 share from his/her perspective?

5. Is there a different way to divide the cake so that each person has a share that they value over \$4? How much will the cake be valued after the new division?

FAIR DIVISION: EQUAL MIGHT NOT BE FAIR

VERSION B

Purpose: This activity reviews the use of fractions, angle measurements, and simple equations. It also shows how a cake can be divided equally in size, but the value of each piece depends on the likes and dislikes of each of the people involved. Note: Chocolate is on the left-hand side; strawberry is on the right-hand side of the cake.



The cake above costs \$12 and is covered in half strawberry and half chocolate frosting. The central angle on each of Abby and Bill's slice is 120°. Abby likes chocolate twice as much as strawberry. Bill likes both frostings the same. Carl likes chocolate three times as much as strawberry. Since the cake cost \$12 total, each person's share should be worth \$4 in their eyes.

1. What is the central angle of Bill's piece? Of the strawberry part?

2. How much is the strawberry half and the chocolate half valued by each person?

3. What is the dollar value of Abby's piece to Abby, Bill's piece to Bill, and Carl's piece to Carl? Justify your answer.

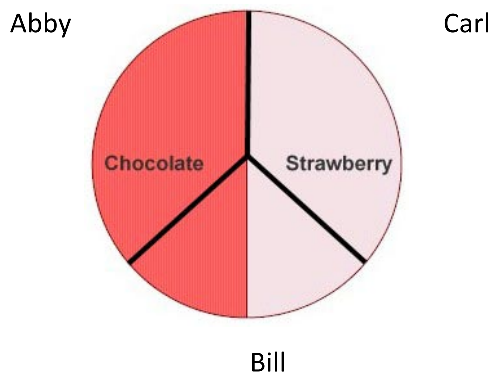
4. Is there an unhappy person? How could one or both of the other two cut their share and/or trade parts with the unhappy person so that each person has a \$4 share from his/her perspective?

5. Is there a different way to divide the cake so that each person has a share that they value over \$4? How much will the cake be valued after the new division?

FAIR DIVISION: EQUAL MIGHT NOT BE FAIR

VERSION C

Purpose: This activity reviews the use of fractions, angle measurements, and simple equations. It also shows how a cake can be divided equally in size, but the value of each piece depends on the likes and dislikes of each of the people involved. Note: Chocolate is on the left-hand side; strawberry is on the right-hand side of the cake.



The cake above costs \$12 and is covered in half strawberry and half chocolate frosting. The central angle on each of Abby and Bill's slice is 120°. Abby likes chocolate but refuses to eat strawberry. Bill likes both frostings the same. Carl likes chocolate three times as much as strawberry. Since the cake cost \$12 total, each person's share should be worth \$4 in their eyes.

1. What is the central angle of Bill's piece? Of the strawberry part?

2. How much is the strawberry half and the chocolate half valued by each person?

3. What is the dollar value of Abby's piece to Abby, Bill's piece to Bill, and Carl's piece to Carl? Justify your answer.

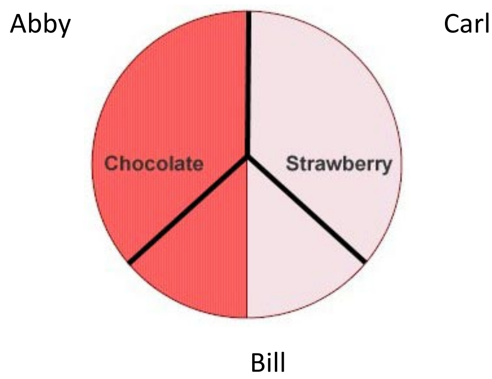
4. Is there an unhappy person? How could one or both of the other two cut their share and/or trade parts with the unhappy person so that each person has a \$4 share from his/her perspective?

5. Is there a different way to divide the cake so that each person has a share that they value over \$4? How much will the cake be valued after the new division?

FAIR DIVISION: EQUAL MIGHT NOT BE FAIR

VERSION D

Purpose: This activity reviews the use of fractions, angle measurements, and simple equations. It also shows how a cake can be divided equally in size, but the value of each piece depends on the likes and dislikes of each of the people involved. Note: Chocolate is on the left-hand side; strawberry is on the right-hand side of the cake.



The cake above costs \$12 and is covered in half strawberry and half chocolate frosting. The central angle on each of Abby and Bill's slice is 120°. Abby likes strawberry twice as much as chocolate. Bill likes strawberry three times as much as chocolate. Carl likes chocolate the same as strawberry. Since the cake cost \$12 total, each person's share should be worth \$4 in their eyes.

1. What is the central angle of Bill's piece? Of the strawberry part?

2. How much is the strawberry half and the chocolate half valued by each person?

3. What is the dollar value of Abby's piece to Abby, Bill's piece to Bill, and Carl's piece to Carl? Justify your answer.

4. Is there an unhappy person? How could one or both of the other two cut their share and/or trade parts with the unhappy person so that each person has a \$4 share from his/her perspective?

5. Is there a different way to divide the cake so that each person has a share that they value over \$4? How much will the cake be valued after the new division?

CUTTING PROCEDURE ANALYSIS WORKSHEET

Description and Sample Answer Key

Divide the class into 6 groups and distribute copies of the Cutting Procedure Analysis worksheet. Each group will be assigned one of the following approaches to fair division: Lone Divider, Last Diminisher, or the Selfridge-Conway Envy-Free Procedure for Three Players. Thus each approach should be worked on by two separate groups. After the groups have done their research and planning together, select one group to present to the class for each of the procedures, then have the other group answer questions from you and from the class.

Description of Procedure: **Lone Divider**

This is a procedure for dividing a cake for three people (A, B, and C), A divides cake into three equal (to him) pieces called X, Y, and Z. Person B and C decide if any of the pieces are of a size or value of at least 1/3 share. If person B approves of X and Y and Person C approves of pieces Y and Z, then we have a division. B can get X and C can get Y and A takes what is left over.

If both B and C only value one piece say X, then give A one of the other pieces say Y. Then X and Z are put back together and B and C then do the one cut and one pick method of sharing that piece.

The problem is that it doesn't extend to more than three players very easily. It is not necessarily envy free either. The reason it might not be envy free is the following. A cuts the cake into three pieces, but both B and C think piece X is the largest. So A gets Y or Z and the other piece is put back with X. Now B and C do the one cuts and one chooses procedure. It is possible that in the opinion of A one of those pieces is more than 1/3 and A will envy whoever gets that piece.

Description of Procedure: **Last Diminisher**

This is procedure for dividing a cake for four or more players (A, B, C, and D) A cuts a piece of cake that she thinks is ¼ or the cake. She passes it on to B. If B thinks it is ¼ of cake or less then he passes it on to C. Suppose C thinks it is larger than 1/4 . He then shaves off a sliver of the cake so that in his view it is now ¼ of the cake. The sliver of cake cut off is put back with the rest of the cake. C then passes the cake onto person D. If D thinks it is less than or equal to ¼ of the cake he passes it on, otherwise it is cut back even farther. The cake is passed in like manner until all people have had a chance to judge it. It is then awarded to the last person who adjusted it, or it goes to A (the person who originally cut it) if everyone agreed it was less than or equal to ¼ .

Once a person gets a piece of cake, he leaves the process and the next cutter starts the process again and it continues until everyone has a piece.

The problem is that the procedure doesn't move to more players very easily because of all the slivers of cake that are handed back to the original. It is messy. It also isn't envy free, although it does give proportional shares.

It is not envy free. An example might be that A cuts what she views as ¼ and everyone lets her have that piece. B cuts next and he cuts a piece that he views as ¼ and everyone lets him have that share. A doesn't have a voice in that awarding and she may think he actually has a piece that is more than ¼.

Description of Procedure: **Selfridge-Conway Envy-Free Procedure for Three Players**

Player A cuts the cake into 3 equal pieces (He views them all the same and would be happy with any of them and not envy any of the other players if they got one of the pieces or a smaller piece than any of the three). He hands the pieces to player B.

Player B judges the three pieces. If he views them all equal he will pass them on, however, if he views one of them too big, he will shave off a piece so that there is now at least a two way tie for the largest piece. He puts the trimmed off piece T aside. He passes on the three pieces, one of which might have been shaved.

Player C chooses a piece. Since he chooses first he is sure to have what he considers at least 1/3 share. He will not envy anyone else. He then hands back the two remaining pieces to Player B.

Player B then chooses. If the trimmed piece comes back to B, he must pick that one. Since he is the one who shaved the piece, he did so in such a way that I was at least one of two pieces tied for the largest. So he doesn't envy anyone else's share. If the shaved piece doesn't come back then there is at least one more piece he considers tied for the largest and he will pick that one. Notice there is no way for the trimmed piece to get to player A (the original cutter, since for sure he wouldn't like it. He thought the pieces were equal to begin with).

Player A gets the left over piece.

No one envies another person's share.

CUTTING PROCEDURE ANALYSIS WORKSHEET

Procedure Name ____

Research the cake cutting procedure you have been assigned and record all of the information that applies in the chart below. Review the chart in groups and develop a plan to teach the rest of the class your system. You will have 10 minutes to teach your concept to the other students in the class. One group will teach the procedure and the other group will answer questions.

Description of Procedure:

Plan for Teaching:

Class Notes **Topic:** ________________ **Questions:**	**Name:** ______________________________ **Class:** ______________________________ **Period:** ______________________________ **Date:** ______________________________ **Notes:**

Summary:

Summary:

CANDY DIVIDING ACTIVITY

Activity Description

PURPOSE: This is an activity to show that there is a procedure Method of Markers that will allow all parties to get what they envision as a fair share plus some more. This procedure works with dividing up a land parcel also.

Method of Markers: A discrete fair division scheme.

In this method the items to be divided are lined up in a row (in our case a row of candies). Each of the players (in our case 6 groups) is given markers of different colors with which to divide the candy into 6 fair portions. Each player must divide the candy into share in such a way that he would be happy with any of the portions. The players lay down their markers independently of what others have done. They put the markers down in a way that none of the other players can see their choices. In our case make copies of how the candy is laid out. Give each group a copy, they place their markers on the paper, and then the teacher puts the colored markers down on the real candy after he has all copies from the groups.

The picture below shows the divisions down by 4 groups, just for the efficiency of showing a smaller problem. X's are candies **R**, **B**, **Y** and **G** are markers.

Group 1 has red markers **R**
Group 2 has blue markers **B**
Group 3 has yellow markers **Y**
Group 4 has green markers **G**

X X X X X **B** **R** X X **G** X **Y** X X X **B** X **G** **R** X **Y** X X X X X **R** **B** X **Y** X **G** X X X X XX

Notice not all the shares have the same number of candies (there are different types of candies and some groups value some types higher than others). Now start at either end and move until you reach the first marker, so starting from the left we move until we hit the blue or red marker. Flip a coin to see who gets those first 5 candies (either red or blue. Let's say Blue gets the vote, Group 2 takes its candies and all blue markers are removed.

{X X X X X, **B**} **R** X X {X **Y** X X X X, **G**} **R** X **Y** X X X X X **R** X **Y** X X X X X XX

The blue candies are gone. We next look for the first set of candies enclosed between two like colored markers. That is the candies that are in green. The two candies between the blue candies and the green candies are extra candies and are set aside. Now green has its share (group 4). All green markers are removed. And the green candies are gone.

{X X X X X,**B**} **R** X X {X **Y** X X X X,**G**} **R** {X **Y** X X X X X,**R**} **R** X **Y** {X X X X X XX,**Y**}

Now starting from the left again, find the first set of candies enclosed between two like colored markers. The first set is the six candies between the two red markers. They are marked in red. Group 1 gets the red candies and they are removed. Yellow markers are left, and group 3 needs its candies. The candy between the last red and yellow marker is extra, since group 3 thought that a fair share would be from the yellow marker to the end (the yellow candies). In this particular cutting, there are three candies left over.

This procedure can be used to divide up property also. You can make a picture of a plot of land with a lake, a house, a barn, and various landscapes. Students can follow the same procedure. Supposedly you can get fair portions by dividing the property into four pieces and the pieces don't have to be divided by parallel lines. One can use odd shaped regions. This might be an extension for some groups to try.

Each group must quietly mark on this oval where it would place its markers so that the whole group would feel that any of the sections would be a fair portion. Indicate the candy that would be on either side of each marker. Hand in your diagram without sharing the placement of your markers with any other group.

If we were now to do the same for the boat (we leave the details to you), the corresponding final settlement would be

Bob	Carol	Ted	Alice
boat − $20,875	$7625	$6625	$6625

Putting the three separate analyses (house, cabin, and boat) together, we get a final settlement of

Bob:	boat + ($40,000 + $22,500 − $20,875 = $41,625)
Carol:	house + (−$140,000 + $17,500 + $7625 = −$114,875)
Ted:	cabin + ($45,000 − $60,000 + $6625 = −$8375)
Alice:	$55,000 + $20,000 + $6625 = $81,625.

Notice that here, Carol gets the house but must pay $114,875 in cash (and Ted gets the cabin but must put up $8375 in cash). This cash is then disbursed to Bob and Alice. In practice, Carol's having this amount of cash available may be a real problem—the key drawback to Knaster's procedure. Nevertheless, Knaster's procedure shows again that whenever some participants have different evaluations of some objects, there is an allocation in which everyone obtains more than what they would normally consider a fair share.

We summarize Knaster's inheritance procedure as follows.

Basic Steps in Knaster's Inheritance Procedure with *n* Heirs PROCEDURE

For each object, the following steps are performed:

Step 1. The heirs—independently and simultaneously—submit monetary bids for the object.

Step 2. The high bidder is awarded the object, and he or she places all but $1/n$ of his or her bid in a kitty. So, if there are four heirs ($n = 4$), then he or she places all but one-fourth—that is, three-fourths—of his or her bid in a kitty.

Step 3. Each of the other heirs withdraws from the kitty $1/n$ of his or her bid.

Step 4. The money remaining in the kitty is divided equally among the n heirs.

13.3 Fair Division and Organ Transplant Policies

In 1984, the United States Congress passed the National Organ Transplant Act and established a unified transplant network known as the Organ Procurement and Transplantation Network (OPTN). One of the primary goals of the OPTN was to increase the equity in the national system of organ allocation.

Achieving an equitable system of organ allocation is complicated by factors other than demand exceeding supply. For example, should an available organ go to the patient who needs it the most or the one for whom the likelihood of a successful transplant is greatest? Should both of these be taken into consideration, and, if so, how? Questions such as these reveal the extent to which an equitable system of organ allocation is a challenging problem in fair division.

In order to illustrate some of the issues (and paradoxes!) arising in the search for an equitable system for organ allocation, we'll (roughly) follow Peyton Young's synopsis—from his book, listed in Suggested Readings—of the fair division procedure for kidney allocation adopted by the OPTN in the late 1980s.

There were three (main) criteria used in arriving at a final ranking of those needing a kidney, and each potential recipient was awarded points according to a fixed method that we now describe.

- **Criterion 1: Waiting time.** A list of potential recipients was made according to how long they had been waiting for an organ. For each potential recipient, one calculates the fraction of people at or below the spot on the list he or she occupies, and then awards that person a number of points equal to 10 times that fraction. So if there are 5 people on the list, the first (waiting the longest) gets $10 \times 1 = 10$ points, the second gets $10 \times (4/5) = 8$ points, the third gets $10 \times (3/5) = 6$ points, and so on.
- **Criterion 2: Suitability.** The donor and potential recipient each have 6 relevant antigens that are either matched or not matched, with the likelihood of a successful transplant increasing with more matches. Two points are awarded for each match.
- **Criterion 3: Disadvantage.** Each person has antibodies that rule out a certain percentage of the population as being potential donors for that person. For some, only 10% are ruled out, while for others it may be as high as 90%. Those in the latter category are at a serious disadvantage compared to those in the former. Thus, potential recipients are awarded 1 point for each 10% of the population they are "sensitized against."

To illustrate this allocation procedure, let's assume we have 5 potential recipients–A, B, C, D, and E–with the following characteristics:

Potential recipient	Months waiting	Antigens matched	Percent sensitized
A	5	2	10
B	4.5	2	20
C	4	0	0
D	2	3	60
E	1	6	90

According to the procedure we described, points would be allocated as follows:

Potential recipient	Months waiting	Antigens matched	Percent sensitized	Total points
A	10	4	1	15
B	8	4	2	14
C	6	0	0	6
D	4	6	6	16
E	2	12	9	23

Thus, if one kidney became available, it would go to E (with 23 points). Presumably, if two became available at the same time, E would get one and D (with 16 points) would get the other.

But now things get interesting. Peyton Young, being well versed in the paradoxes of voting theory, fair division, and apportionment (among other things), observed the following. In the above scenario, what if two kidneys become available, but one is delayed slightly? Presumably, E gets the first one, and then we redo the chart with only A, B, C, and D. This yields the following:

Potential recipient	Months waiting	Antigens matched	Percent sensitized
A	5	2	10
B	4.5	2	20
C	4	0	0
D	2	3	60

According to the procedure we described, points would be allocated as follows:

Potential recipient	Months waiting	Antigens matched	Percent sensitized	Total points
A	10	4	1	15
B	7.5	4	2	13.5
C	5	0	0	5
D	2.5	6	6	14.5

Thus, A (not D!) now gets the second kidney, having 15 points to 14.5 for D. This is an example of what is called the "priority paradox." For more on this, we invite the reader to consult Peyton Young's book in the Suggested Readings.

13.4 Taking Turns

For many of us, an early lesson in fair division happens in elementary school with the choosing of sides for a spelling bee or when picking teams on the playground. In terms of importance, these pale in comparison with the issue of property settlement in a divorce. Remarkably, however, the same fair-division procedure—*taking turns*—is often used in both.

Taking turns is fairly self-explanatory. With two parties (and that's all we'll consider here), one party selects an object, then the other party selects one, then the first party again, and so on. But in this context, there are several interesting questions that suggest themselves:

1. How do we decide who chooses first?
2. Because choosing first is often quite an advantage, shouldn't we compensate the other party in some way, perhaps by giving him or her extra choices at the next turn?
3. Should a player always choose the object he or she most favors from those that remain, or are there strategic considerations that players should take into account?

The answer to question 1 is often "toss a coin," but there are other possibilities—for example, the two parties could "bid" for the right to go first, as in an auction. The answer to question 2 is less clear, but we outline a discussion of the issue it raises in Writing Project 2.

Question 3, on the other hand, is remarkably interesting, and it is this one that we want to pursue. Let's look at an easy example. Suppose that Bob and Carol are

Fair Division and Organ Transplant Policies

1. Construct the table showing how points would be allocated among the following four potential recipients for a kidney transplant, according to the scheme on 7e / 8e pp. 413-415.

Potential recipient	Months waiting	Antigens matched	Percent sensitized
A	9	2	20
B	6	3	0
C	5	4	40
D	2	6	60

2. Does the example in #1 give rise to the same kind of paradox as in 8e pp. 413-415? Explain why or why not.