Module 11: Game Theory

This material corresponds to chapter 15 of the textbook
For All Practical Purposes
GAME THEORY
CHAPTER 15

TIME FRAME: 12 days

ENDURING UNDERSTANDINGS:

Mathematics provides tools to analyze situations involving conflict and cooperation between two or more parties.

ESSENTIAL (ASSESSMENT) QUESTIONS:

1. What are the differences between a zero sum game and a non-zero sum game?
2. How do we find optimal strategies for a mixed strategy zero sum game (2 by 2 matrix)? How would we do this if the opponents have more than 2 strategies? (Optional)
3. What are the differences between the games of chicken and prisoners dilemma? (How do the Nash Equilibrium compare?)

CRMS:

PS/Reasoning: 1.1, 1.2, 1.3
Communication: 2.1, 2.2, 2.3
Connections: 3.1, 3.4
Algebra: 7.1, 7.3
Probability and Statistics: 6.1, 6.4

AT THE END OF THE MODULE STUDENTS WILL KNOW AND BE ABLE TO:

1. Solve saddle-point or pure strategy games.
2. Solve 2 x 2 nonsaddle-point or mixed strategy games.
3. Recognize dominant (or dominated) rows and columns in game matrices
4. Reduce some dames of larger dimensions to solvable form (optional)
5. Analyze non-zero sum games.
6. Model simple real life situations with game theory.

PREREQUISITE KNOWLEDGE/SKILLS: Expected value, probability, linear functions

PRE-ASSESSMENT: None
ACTIVITIES

Introductory problem, **WWII Battle** from HiMAP Module 3: “The Mathematics of Conflict”, p. 3

**Odd/Even Game**, p.3 HiMAP Module
Worksheet: **Exploring Mixed Strategies**
Worksheet: **Playing a Mixed Strategy Game**
Worksheet: **Analyzing a Mixed Strategy Game**
Worksheet calculator activity: **Discovering Fair Games**

*Numb3rs* activity: To C or Not To C
*Numb3rs* video clip (From Project TIME DVD)
Worksheet: **Topology of Non-Zero Sum Games**
Worksheet: **Interesting Non-Zero Sum Games**
*Footloose* video clip (From Project TIME DVD)

POST-ASSESSMENT:

- Paper-pencil test
- Group analysis of Real Life Situation
  - *Optional*: Socratic Seminar of Reading – Forgiveness Math

RESOURCES

HiMAP Module: **The Mathematics of Conflict**
Video Clip from “*Numb3rs*”
Video Clip from “*A Beautiful Mind*”
Video Clip from “*Footloose*”
Article from Discover Magazine: **Forgiveness Math**
  http://discovermagazine.com/1993/may/forgivenessmath212
Article from Whitman Magazine: **Given the odds, ethics, would you go for the extra credit?**
  http://www.whitman.edu/magazine/march2008/campusnews.pdf
Article from Discover Magazine: **Darwin plays Game Theory and wins**
Reading Anticipation Guides
Optional readings from text: These readings will replace the direct instruction on vocabulary and methods students will preview before reading. They will take Cornell Notes from the text. Assign the readings BEFORE the respective day.

Day 1: Pure Strategy Games
7e pp. 542-548 / 8e pp. 467-472

Day 2: Mixed Strategy Games
7e pp. 550-553 / 8e pp. 474-476

Day 3: Mixed Strategy Matrices
7e pp. 554-555 / 8e pp. 477-478

Day 4: Analyzing Mixed Strategy Games
7e pp. 556-559 / 8e pp. 478-480

Day 5: Analyzing Mixed Strategy Games
7e pp. 559-560 / 8e pp. 481-482

Day 7: Non Zero Sum Games: Prisoner’s Dilemma
7e pp. 560-564 / 8e pp. 482-485

Day 8: Chicken
7e pp. 565-566 / 8e pp. 486-487

Day 1
• Introductory Problem: WWII Battle.
• Hand out Page 3 of HiMAP Module.
• Group Discussion of Strategies of Commanders then Share Out.
• Definition of pure strategy (dominant strategy), value of game, Nash Equilibrium, zero sum game, minimax theorem. Students take Cornell Notes.

HW: Newspaper assignment: Find articles in local or national paper that involves conflict or cooperation in the competition for ideas, money, respect, resources, or survival. In other words find a game. (Students may need more than one night to complete this assignment.)

Day 2
• Go over articles, discuss whether one winner and one loser or possibility of win-win or lose-lose situations, Sort into zero and non-zero sum games.
• Students play Odd/Even Game on page 3 of HiMAP Module twice and analyze and share out ideas about game.
• Introduce Vocabulary of mixed strategy zero-sum game.
• If time, have students play Odd/Even game again 20 times this time each person playing each strategy half of the time and then calculating the winner. Purpose here is to show that even with an optimal strategy one may lose.

Day 3
• Go over Concept Map and big picture about games.
• Direct instruction on rational players, saddle point, dominant row or column, dominated strategy, pure strategy, value of game.
• Analyze a sample 3 by 2 pure strategy game with both dominant row or column method and with minimax method. Find saddlepoint, Nash equilibrium.
• Play Rock-Paper-Scissors and analyze.
HW: 7e p. 584 #1-8 / 8e p. 502 #1-8

Day 4
• Go over homework.
• Handout worksheet: Exploring Mixed Strategies, Students work in pairs
• Students share out ideas about value of game, minimax method, expected value and finding optimal strategies
• Handout worksheet: Playing a Mixed Strategy Game, Students play game
• Teacher direct instruction on analyzing game with expected value and with graphing
• Handout worksheet: Analyzing Mixed Strategy Game
HW: Students analyze Mixed Strategy game from Y’s perspective.

Day 5
• Go over homework.
• Finding optimal strategy with expected values and graphing.
HW: 7e p. 584 #9, 10, and 16 / 8e p. 502 #9, 10, and 16

Day 6
• Go over homework.
• Practice graphing method for finding optimal strategies with 2X2 matrix.
• Download Graphing calculator program.
HW: Worksheet: Discovering Fair Games

Day 7
• Go over homework.
• Introduce Non-Zero Sum Games, if possible use some of the examples brought into class. Introduce “Prisoners Dilemma” specifically, and follow dominant row, column method to find Nash Equilibrium point.
• Video Clip Numb3rs video clip: “To C or Not to C”.
• In-class activity from Numb3rs: To C or Not to C.
HW: 7e p. 587 #22-24 / 8e p. 504 #22-24 and read the articles from Discover Magazine and Whitman Magazine.
Day 8
- Go over homework.
- Video Clip: “Footloose”.
- Direct Instruction on game of Chicken.
- Other Interesting Games (2X2 non-zero sum games, Leader, Battle of Sexes)
- Show video clips from “A Beautiful Mind: The dance scene” and “Footloose: The librarian”.

HW: Groups look over examples and chose one for Team Paper on Real life example and answer questions from Discover article.

Day 9
- In-class Paper/Pencil Test.
- Turn in team paper (1 page) with write-up about Footloose or A Beautiful Mind clip.

Optional: If you have a full day that can be added to the module you can have a Socratic Seminar on Forgiveness Math article.
INTRODUCTORY PROBLEM

Description

The following page is designed to be copied onto a transparency to be used with an overhead projector. It summarizes the key choices of two military commanders in a battle from World War II, which is used to introduce game theory to students. Based on the story below, the instructor should describe the situation to students using the transparencies as a visual reference. Then ask the students, based on the data summarized in the matrix, what decision each commander should make. Students should discuss their reasoning in small groups, then share out with the instructor moderating. If possible, the instructor should lead the discussion toward the correct game-theoretic solutions without explicitly telling the students what the answer is. Many classes will come up with the correct answer very quickly, while others might need a little more guidance in the form of an instructor asking them to explain their reasoning and asking “What if” questions if the students propose incorrect solutions.

THE STORY

Read Section 1 of HiMAP Module 3, “The Mathematics of Conflict” from the HiMAP CD-ROM, and summarize the events of the battle described for your students. This story describes a real-life situation from World War II, and it illustrates the kind of strategic thinking that is essential to game theory.
Module 11: Game Theory

General Kenney

Search North

Search South

Japanese Commander

Sail North

Sail South

2 days

2 days

1 day

3 days
### Unit Homwork Schedule

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### Unit Self Test Questions

1. What are the differences between a zero sum game and a non-zero sum game?
2. How do we find optimal strategies for a mixed strategy zero sum game (2 by 2 matrix)? How would we do this if the opponents have more than 2 strategies? (Optional)
3. What are the differences between the games of chicken and prisoners dilemma? (How do their Nash Equilibrium compare?)

### Unit Vocabulary

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Rational choice</th>
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<tbody>
<tr>
<td>Pure strategy</td>
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<tr>
<td>Mixed strategy</td>
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<tr>
<td>Zero sum game</td>
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<tr>
<td>Minimax</td>
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<tr>
<td>Saddlepoint</td>
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<tr>
<td>Value of game</td>
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<td>Expected value</td>
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<td>Dominance</td>
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</table>
### Vocabulary Preview

<table>
<thead>
<tr>
<th>Terms</th>
<th>??</th>
<th>?</th>
<th>!</th>
<th>Written Definition</th>
<th>Logograph</th>
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</thead>
<tbody>
<tr>
<td>Game Theory</td>
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<tr>
<td>Game Strategies</td>
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<tr>
<td>Rational Choice</td>
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<td>Total Conflict</td>
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<td>Partial Conflict</td>
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<td>Maximin Strategy</td>
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<td>Minimax Strategy</td>
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<td>Saddlepoint</td>
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<td>Value</td>
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</table>

**Key:**
- ?? I have NO idea what this means
- ? I have heard it before…but I’m not sure
- ! I know this word! It means…

**Logograph:** Sketch what your mind “sees” when you read each word.
**ANTICIPATION GUIDE**

**INSTRUCTIONS:**
Read each statement and write **Agree** in the blank if you believe the statement and could support it or put **Disagree** in the blank if you do not believe it or could not support it. After you finish reading the selection – we will revisit this and check the validity of each statement.

<table>
<thead>
<tr>
<th>Before Reading</th>
<th>After Reading</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Game Theory</strong> is the mathematics of competition.</td>
<td></td>
</tr>
<tr>
<td>Game Theory cannot be applied to real-world situations.</td>
<td></td>
</tr>
<tr>
<td>The <strong>maximin</strong> is the minimum value of a collection of maximum values of a game situation.</td>
<td></td>
</tr>
<tr>
<td>The game of tic-tac-toe does not have a <strong>saddlepoint</strong>.</td>
<td></td>
</tr>
</tbody>
</table>
# 15.2 Mixed Strategies


## Vocabulary Preview

<table>
<thead>
<tr>
<th>Terms</th>
<th>??</th>
<th>?</th>
<th>!</th>
<th>Written Definition</th>
<th>Logograph</th>
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</thead>
<tbody>
<tr>
<td>Pure Strategy</td>
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<tr>
<td>Mixed Strategy</td>
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<td>Expected Value</td>
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</table>

**Key:**
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<table>
<thead>
<tr>
<th>Before Reading</th>
<th>After Reading</th>
</tr>
</thead>
<tbody>
<tr>
<td>The most competitive games usually do not have a saddlepoint.</td>
<td></td>
</tr>
<tr>
<td>A <strong>pure strategy</strong> is a special case of a mixed strategy that assigns the probability of 1/2 to each of the pure strategies.</td>
<td></td>
</tr>
<tr>
<td>The <strong>expected value</strong> of a game is the average value.</td>
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</tbody>
</table>
# 15.2 Mixed Strategies


## Vocabulary Preview

<table>
<thead>
<tr>
<th>Terms</th>
<th>??</th>
<th>?</th>
<th>!</th>
<th>Written Definition</th>
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<tbody>
<tr>
<td>Payoff Matrix</td>
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<tr>
<td>Mixed Strategy</td>
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<td>Value</td>
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<tr>
<td>Fair Game</td>
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<table>
<thead>
<tr>
<th>Before Reading</th>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>A payoff matrix</strong> represents the earnings that Keanu Reeves made while playing the character Neo.</td>
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</tr>
<tr>
<td><strong>A game is fair</strong> if you always have a chance to win.</td>
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</tr>
</tbody>
</table>
15.2 NONSYMMETRICAL GAMES
7e Text: pp. 556-559 / 8e Text: pp. 478-480

Vocabulary Preview

<table>
<thead>
<tr>
<th>Terms</th>
<th>??</th>
<th>?</th>
<th>!</th>
<th>Written Definition</th>
<th>Logograph</th>
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</thead>
<tbody>
<tr>
<td>Nonsymmetrical Game</td>
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<tr>
<td>Symmetrical Game</td>
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<tr>
<td>Zero-Sum Game</td>
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<table>
<thead>
<tr>
<th>Before Reading</th>
<th>After Reading</th>
</tr>
</thead>
<tbody>
<tr>
<td>The solution to a system of linear equations can be used to represent the expected value of the game for a player when using their optimal strategy.</td>
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<tr>
<td><strong>Nonsymmetrical games</strong> are an example of <strong>zero-sum games.</strong></td>
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<tr>
<td>In <strong>zero-sum games</strong> both players could win.</td>
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</tbody>
</table>
# 15.2 Zero Sum Games
7e Text: pp. 559-560 / 8e Text: pp. 481-482

## Vocabulary Preview

<table>
<thead>
<tr>
<th>Terms</th>
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<th>Written Definition</th>
<th>Logograph</th>
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<tbody>
<tr>
<td>Minimax Theorem</td>
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<table>
<thead>
<tr>
<th>Before Reading</th>
<th>After Reading</th>
</tr>
</thead>
<tbody>
<tr>
<td>The <strong>minimax theorem</strong> states that there is no guarantee that there will be a unique game value and optimal strategy for each player.</td>
<td></td>
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</tbody>
</table>
Vocabulary Preview

<table>
<thead>
<tr>
<th>Terms</th>
<th>??</th>
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<th>!</th>
<th>Written Definition</th>
<th>Logograph</th>
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<tr>
<td>Variable Sum Games</td>
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<td>Partial Conflict Games</td>
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<td>Prisoner’s Dilemma</td>
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<td>Dominant Strategy</td>
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<td>Nash Equilibrium</td>
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<table>
<thead>
<tr>
<th>Before Reading</th>
<th>After Reading</th>
</tr>
</thead>
<tbody>
<tr>
<td>In the <strong>Prisoner’s Dilemma</strong>, there are 2 strategies: to escape prison or to remain in prison.</td>
<td></td>
</tr>
<tr>
<td>In the <strong>Prisoner’s Dilemma</strong>, it is always better to cooperate.</td>
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</tr>
<tr>
<td>The <strong>Prisoner’s Dilemma</strong> is an example of a total conflict game.</td>
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<tr>
<td>A <strong>Nash Equilibrium</strong> is established when no player benefits from changing his/her strategy.</td>
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</table>
# 15.3 Partial Conflict Games


## Vocabulary Preview

<table>
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<tr>
<th>Terms</th>
<th>??</th>
<th>?</th>
<th>!</th>
<th>Written Definition</th>
<th>Logograph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game of Chicken</td>
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<tr>
<td>Digraph</td>
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**Anticipation Guide**

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<table>
<thead>
<tr>
<th>Before Reading</th>
<th>After Reading</th>
</tr>
</thead>
<tbody>
<tr>
<td>In the <strong>game of chicken</strong>, there are 2 strategies: to swerve or not to swerve.</td>
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<tr>
<td>The best strategy of chicken is to <strong>always</strong> swerve.</td>
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</table>
EXPLORING MIXED STRATEGIES

Description and Answer Key

**Estimated Time:** 30 minutes.

**Description**

The worksheet on the next 4 pages is designed to allow students to explore the meaning of a mixed strategy. There are times when playing a single strategy every time (called a pure strategy) will allow an opponent to take advantage of you, but a mixed strategy, where you switch between various strategies, can improve your results. (For example, you could play Strategy A one third of the time and Strategy B two thirds of the times.) The purpose of such a strategy is to make it harder for an opponent to take advantage of your choices.

**Solutions**

1) I should play A since I would win $2 instead of losing $1.

2) I should play B because I would win $4 instead of losing $6.

3) 
   a. I would get 2 half of the time and -6 half of the time, so
      \[ \frac{1}{2} (2) + \frac{1}{2} (-6) = 1 - 3 = -2 \]
      I would lose $2 on average.
   b. \[ \frac{1}{2} (-1) + \frac{1}{2} (4) = -1/2 + 2 = 1.5 \]
      I would win $1.50.
   c. Strategy B is better. I would win not lose with that strategy.

4) If I play A
   \[ \frac{3}{4} (2) + \frac{1}{4} (-6) = 6/4 - 6/4 = 0 \]
   I would not win or lose any money. If I play B
   \[ \frac{3}{4} (-1) + \frac{1}{4} (4) = -3/4 + 4/4 = 1/4 \]
   I would win $0.25. I should play B because I would win $0.25.
5) If I play A
\[
\frac{2}{3} (2) + \frac{1}{3} (-6) = 4/3 - 6/3 = -2/3
\]
I would lose $0.67. If I play B
\[
\frac{2}{3} (-1) + \frac{1}{3} (4) = -2/3 + 4/3 = 2/3
\]
I would win $0.67. I should play B.

6) She should play B. She would win $6.

7) She should play A. She would win $1.

8) If Y plays A
\[
\frac{1}{2} (2) + \frac{1}{2} (-1) = 2/2 - 1/2 = 1/2
\]
Y would lose $0.50. If Y plays B
\[
\frac{1}{2} (-6) + \frac{1}{2} (4) = -6/2 + 2/2 = -4/2
\]
Y would win $2. My opponent should play B since she would win.

TEACHER NOTES: Be sure to summarize this assignment after students have finished it. The primary thing we want to show is that no matter what strategy my opponent chose in problems 1 through 5, I could manage my choice so that I would win more. If she had chosen just the right percentage of the time playing each strategy A and B, then I would not be able to manipulate the outcome and it would be set. That outcome would be the value of the game. Similarly, in problems 6 through 8, no matter what I chose, my opponent could manipulate his choices to be more beneficial to him. If I had chosen the right percentage for strategy A or B then there would be no room for manipulation.
You are playing the following 2x2 game with an imaginary opponent. It is a repeated game – you play over and over until someone is out of money. The numbers in the table indicate payoffs to you: if the entry is positive, that indicates money that your opponent pays to you; if the entry is negative, that indicates money you pay to your opponent.

<table>
<thead>
<tr>
<th></th>
<th>Strategy A</th>
<th>Strategy B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy A</td>
<td>2</td>
<td>-6</td>
</tr>
<tr>
<td>You</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>Strategy B</td>
<td>-6</td>
<td>4</td>
</tr>
</tbody>
</table>

(1) If you know that your imaginary opponent always plays his strategy A, what strategy should you play? Why?

(2) If you know that your opponent always plays his strategy B, what strategy should you play? Why?
Suppose that your imaginary opponent plays his strategy A half of the time and strategy B the other half of the time. You cannot tell in advance which strategy he will play, only that each option is just as likely for him.

a. What will your expected payoff be if you play strategy A? *(Hint: Recall the idea of expected value from probability.)*

b. What if you play strategy B?

c. Which one is better? Explain.
(4) Suppose that your imaginary opponent plays his strategy A three-fourths of the time, and he plays his strategy B one-fourth of the time. Now which strategy should you use? Explain.

(6) If you decide to play your strategy A all the time, what should your imaginary opponent do?

(7) If you decide to play your strategy B all the time, what should your imaginary opponent do?

(8) If you decide to play each of your strategies half of the time, what should your opponent do?
PLAYING A MIXED STRATEGY GAME

Description

ESTIMATED TIME: 20 Minutes

DESCRIPTION

The purpose of this activity is to continue on with the analysis of a mixed strategy game that was started in the previous activity. Using what they have learned about this game in that activity, students should play the game 20 times. They should keep track of the “money” they win in their score column. If they didn’t win, they could put down how much they lost (with opposite sign). At the end of twenty rounds, have students calculate their total winnings or losses, and the average win or loss. Have students calculate the p and q values for the game. A class record should be kept of results.
**Playing A Mixed Strategy Game**

Play twenty turns of this game:

$$
\begin{array}{c|c|c|}
X & \text{Choice} & Y & \text{Choice} \\
\hline
a & 2 & b & -6 \\
\hline
b & -1 & 4 & \\
\end{array}
$$

Keep score with a running total.

<table>
<thead>
<tr>
<th>Turn</th>
<th>X+ Choice</th>
<th>Y- Choice</th>
<th>Running Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
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<td>3</td>
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<tr>
<td>20</td>
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</tr>
</tbody>
</table>
ANALYZING A ZERO-SUM MIXED STRATEGY

Description and Guide for Direct Instruction

**ESTIMATED TIME:** 30 minutes (15 min. for direct instruction, 15 min. for worksheet)

The following activity is designed in two parts. The first part is direct instruction, with the teacher going through an example of calculating an optimal mixed strategy for one player in a 2x2 game. The second part is a worksheet which asks the students to calculate the optimal mixed strategy for the other player in the same game.

**DIRECT INSTRUCTION**

Draw the following game on the board or overhead:

```
  Y
 a  b
X
 a  2  -6
 b -1  4
```

Point out that the game has no saddle point, and therefore the optimal strategies will be mixed strategies rather than pure strategies. Then go through the following calculations.

Let p represent the fraction of the time player X will play strategy A, and let q represent the fraction of the time player Y will play strategy A.

```
  Y
q  1-q
a  b
X
 p a  2  -6
1-p b -1  4
```
Player X’s Analysis

A. What is the expected value of the game when Y plays strategy a?
   \[ E(a) = 2p - 1(1-p) \text{ or } 3p - 1 \]

B. What is the expected value of the game when Y plays strategy b?
   \[ E(b) = -6p + 4(1-p) \text{ or } -10p + 4 \]

C. Determine p by solving the equation:
   \[ 3p - 1 = -10p + 4 \]
   \[ p = \frac{5}{13} \approx 0.38 \]

D. Determine the value of the game by substituting the value of p into either of the expected value equations.
   \[ V = .15 \text{ (thus the game in X’s favor.)} \]

SOLUTIONS TO WORKSHEET QUESTIONS

Player Y’s Analysis

A. What is the expected value of the game when X plays strategy a?
   \[ E(a) = 2q - 6(1-q) = 8q - 6 \]

B. What is the expected value of the game when X plays strategy b?
   \[ E(b) = -1q + 4(1-q) = -5q + 4 \]

C. Determine q by solving the equation:
   \[ 8q - 6 = -5q + 4 \]
   \[ q = \frac{10}{13} \approx 0.77 \]

D. Determine the value of the game by substituting the value of q into either of the expected value equations.
   \[ V = .16 \text{ and so the game is in X’s favor (because of rounding error, this doesn’t quite match V from X’s perspective – if students work in fractions the whole time, they won’t encounter such a error)} \]
ANALYZING A MIXED STRATEGY

Calculate the optimal strategies and the value of the game.

\[
\begin{array}{cc}
X & a & b \\
\hline
a & 2 & -6 \\
b & -1 & 4 \\
\end{array}
\]

Since the game has no saddle point and therefore is not a pure game, both players will adopt mixed strategies. Let \( p \) represent the fraction of the time player \( X \) will play strategy \( a \), and let \( q \) represent the fraction of the time player \( Y \) will play strategy \( a \).

\[
\begin{array}{cc}
Y & a & b \\
\hline
q & 1-q \\
\end{array}
\]

\[
\begin{array}{cc}
X & a & b \\
p & 2 & -6 \\
1-p & -1 & 4 \\
\end{array}
\]

Player \( Y \)'s Analysis

A. What is the expected value of the game when \( X \) plays strategy \( a \)?

B. What is the expected value of the game when \( X \) plays strategy \( b \)?

C. Determine \( q \) by solving the equation from setting these expected value equal to each other.

D. Determine the value of the game by substituting the value of \( q \) into either of the expected value equations.
DISCOVERING A FAIR GAME

Description and Answer Key

Estimated Time: 30 minutes (plus setup entering a calculator program)

The worksheet on the following pages asks students to explore the mixed strategies of a general 2x2 game

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

and to determine whether such games are fair, as a function of A, B, C and D, by testing out several different values and quickly checking the qualities of the game via a program on the graphing calculator. The program is included on the reverse side of the worksheet. If the teacher has the ability to easily copy to program onto the students’ graphing calculators, she can do so; otherwise, the students should be asked to carefully enter the program manually for homework. (Note that electronically copying the program will eliminate the need to debug the programs in class, which makes doing so the superior option if is available.) The program is available on the Project TIME CD-ROM.

Solutions

3) Find the product of A and D and the product of B and C. Whichever product is bigger will determine in whose favor the game is. Example: A = -4, B = 6, C = 8, and D = -13, the game is in Y’s favor since the product of A and D is larger than product of B and C. If D = -8, then the game would be in X’s favor since B * C is larger than A * D. If the products are the same then the value of the game is 0.

4) Have students try different cases. Sometimes the results of the computer program are confusing (especially if we remember that the largest value of p is 1 or 100%) Part of the confusing results happen because the game might not be a mixed strategy game at all but a pure strategy game.
5) Three examples that might be of interest to discuss with the class are:

1. \( A = 5, B = 3, C = 4, \text{ and } D = -6 \) (Pure strategy, X should always play A, his dominant row, and therefore Y should always play B (look for Nash Equilibrium)

2. \( A = 4, B = 3, C = 5, \text{ and } d = -6 \) (Pure strategy, Y’s dominant column is B, therefore X should always play A) Look for Nash Equilibrium

3. \( A = 4, B = 6, C = 5, \text{ and } D = -6 \) (Mixed strategy, although it looks like it might be pure strategy game with X always playing A, however, if one looks for Nash Equilibrium we see that there isn’t a dominant row or column, and is a mixed strategy game.)

6) The answer here varies. Usually one should play A most often, and this is counter-intuitive for most students. Have students try

1. \( A = 4, B = -1, C = -20, \text{ and } D = 5 \) (Value of game is 0. Find p)

2. \( A = 1, B = -1, C = -20, \text{ and } D = 20 \) (Again value is 0, find p).

3. \( A = 20, B = -1, C = -20, \text{ and } D = 1 \) (Again value is 0, find p)

Challenge for students: What if game isn’t fair? Does that affect p values?
Discovering a Fair Game

Purpose: This discovery activity allows students to find the relationship between matrix payouts and the value of a game.

1. Copy or transfer the calculator activity into a graphing calculator.

2. Practice inputting different games and finding out if they are fair (i.e. if the value is zero).

3. What is the relationship between A, B, C and D values and a fair game if A and D are both negative and B and C are both positive.

4. In whose favor is the game if A, B, and C are all positive and D is negative?

5. In the case described above, will that game be a pure strategy game? Give examples of why or why not. (Recall the Minimax Theorem and Nash Equilibrium.)

6. Suppose A and D are both negative, and B and C are both positive. If B=1 and C=20, should X play row 1 or row 2 more often? Give examples to justify your answer.

7. Summarize the things that you have learned about 2x2 games using the calculator program.
PROGRAM: GAMETHER
:Lbl 2
:ClrHome
:Disp "ENTER A"
:Input A
:Disp ""
:Disp "ENTER B"
:Input B

:Disp ""
:Disp "ENTER C"
:Input C
:Disp ""
:Disp "ENTER D"
:Input D

:ClrHome
:Lbl 1
:ClrDraw
:DrawF (A−C)*X+C
:DrawF (B−D)*X+D

:Pause
:ClrDraw
:DrawF (A−B)*X+B
:DrawF (C−D)*X+D
:Pause

:DispGraph
:ClrHome
:Disp "P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"

:Disp “ENTER 1 TO REVIEW GRAPHS”
:Disp “ENTER 5 TO END ANALYSIS”
:Input R
:If R=1

:Goto 1
:ClrHome
:Disp ""
:Disp "ENTER 1 TO"
:Disp Graph
:ClrHome
:Disp “P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"

:Disp “ENTER 1 TO REVIEW GRAPHS”
:Disp “ENTER 5 TO END ANALYSIS”
:Input R
:If R=1

:Goto 1
:ClrHome
:Disp ""
:Disp "ENTER 1 TO"
:Disp Graph
:ClrHome
:Disp “P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"

:Disp “ENTER 1 TO REVIEW GRAPHS”
:Disp “ENTER 5 TO END ANALYSIS”
:Input R
:If R=1

:Goto 1
:ClrHome
:Disp ""
:Disp "ENTER 1 TO"
:Disp Graph
:ClrHome
:Disp “P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"

:Disp “ENTER 1 TO REVIEW GRAPHS”
:Disp “ENTER 5 TO END ANALYSIS”
:Input R
:If R=1

:Goto 1
:ClrHome
:Disp ""
:Disp "ENTER 1 TO"
:Disp Graph
:ClrHome
:Disp “P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"

:Disp “ENTER 1 TO REVIEW GRAPHS”
:Disp “ENTER 5 TO END ANALYSIS”
:Input R
:If R=1

:Goto 1
:ClrHome
:Disp ""
:Disp "ENTER 1 TO"
:Disp Graph
:ClrHome
:Disp “P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"

:Disp “ENTER 1 TO REVIEW GRAPHS”
:Disp “ENTER 5 TO END ANALYSIS”
:Input R
:If R=1

:Goto 1
:ClrHome
:Disp ""
:Disp "ENTER 1 TO"
:Disp Graph
:ClrHome
:Disp “P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"

:Disp “ENTER 1 TO REVIEW GRAPHS”
:Disp “ENTER 5 TO END ANALYSIS”
:Input R
:If R=1

:Goto 1
:ClrHome
:Disp ""
:Disp "ENTER 1 TO"
:Disp Graph
:ClrHome
:Disp “P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"

:Disp “ENTER 1 TO REVIEW GRAPHS”
:Disp “ENTER 5 TO END ANALYSIS”
:Input R
:If R=1

:Goto 1
:ClrHome
:Disp ""
:Disp "ENTER 1 TO"
:Disp Graph
:ClrHome
:Disp “P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"

:Disp “ENTER 1 TO REVIEW GRAPHS”
:Disp “ENTER 5 TO END ANALYSIS”
:Input R
:If R=1

:Goto 1
:ClrHome
:Disp ""
:Disp "ENTER 1 TO"
:Disp Graph
:ClrHome
:Disp “P VALUE IS"
:Disp (D−C) / (A−C−B+D) → P
:Disp P

:Disp “Q VALUE IS"
:Disp (D−B) / (D−B+ A−C) → Q
:Disp Q
:Disp “"

:Disp “GAME VALUE IS"
:Disp P*(A−C)+C→ V
:Disp V
:Pause
:ClrHome
:Disp “"
NUMB3RS ACTIVITY: TO C OR NOT TO C

Description and Answer Key

Estimated Time: 15-20 minutes

The following activity is taken from http://education.ti.com/go/NUMB3RS.

The instructor should begin by showing the clip of the episode “The Art of Reckoning” from the Project TIME DVD. This clip demonstrates the idea of the Prisoner’s Dilemma – a model problem in game theory.

Answers

1. Answers will vary. For example, one player might defect every time, hoping to maximize points but change when the player realizes that it does not work.

2. Answers will vary. For example, it may be better to take any punishment rather than risk the death penalty – death could be thought of as having a value as bad as negative infinity.

3. Answers will vary. In general, tit-for-tat will probably do better. It is also possible that one of the players came up with “tit-for-tat” in their first game.

4. See 3 above.

5. The game will be a tie, but one that maximizes the “common good”.

6. Player A wins if the game goes an even number of turns longer; otherwise it is a tie.

7. Both players start with “tit-for-tat.” Then, one player switches to play just C. After playing C for several turns, switch to D and use if for the remainder of the game. This will ensure a (small) win.
NUMB3RS Activity: To C or Not to C

In “The Art of Reckoning,” a convicted killer offers to give the FBI a detailed confession if they will let him visit his daughter. Don wants to pry information from him without giving too much in return. Charlie suggests that they use a game theory strategy “tit-for-tat,” from the game known as Prisoner’s Dilemma.

Prisoner’s Dilemma gets its name from the hypothetical case of two criminals who are arrested and held separately from each other. The police have enough information to indict them on a smaller crime, but suspect they are behind a larger crime. Each is told that if one testifies against the other, he or she will get a sentence less than that for the smaller crime, while the other will get a longer sentence for the larger crime. In Prisoner’s Dilemma, “cooperate” means that the prisoner does not give up the other prisoner; “defect” means that the prisoner does give up the other prisoner. These terms indicate that the prisoners cooperate with each other or defect from one another, not that they cooperate with the investigators.

In an actual case, each prisoner would get only one chance. But in theory, game theorists have made it into a game where the possible outcomes have point values. Play continues for a number of rounds, and the player with more total points wins. There are many strategies for playing, but nobody has ever proven that one strategy is the best. This activity provides the opportunity to experiment with a strategy.

In this activity, the payoff is given as points (not years in prison), so the object is to get as many points as possible. This table shows the point values:

<table>
<thead>
<tr>
<th>Player B</th>
<th>Cooperates</th>
<th>Defects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperates</td>
<td>A gets 3</td>
<td>A gets 1</td>
</tr>
<tr>
<td>Player A</td>
<td>B gets 3</td>
<td>B gets 1</td>
</tr>
<tr>
<td>Defects</td>
<td>A gets 5</td>
<td>A gets 1</td>
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<tr>
<td></td>
<td>B gets 1</td>
<td>B gets 1</td>
</tr>
</tbody>
</table>

Each player needs a copy of this chart:

<table>
<thead>
<tr>
<th>TURN</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
<tr>
<td>A</td>
<td>C</td>
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Each player should secretly decide on a strategy. For each turn, select either C (cooperate) or D (defect) by circling the appropriate letter so that the opponent cannot see the chart, and there is no clue from hearing how the letter is written. Reveal the choices simultaneously and decide the points from the payoff table. You should also record your other opponent’s choice in order to identify his/her strategy.
If a player knows when the game will end, there is an advantage because his/her strategy could be changed for the last turn. It is important that the game is ended by an outsider. Therefore, your teacher or a classmate will tell you when the game ends.

1. Play the above game with a partner before reading on. The teacher will announce when to end the game (the table allows for 20 turns, but the number can be adapted as needed). Discuss the strategies used.

2. How could the strategy be changed for a different set of point values? For example, what value could be given to the cooperating player if when the other player defects, the cooperating player has all points taken away? Why?

Charlie points out that in competition, a rather simple strategy has been used to defeat some of the most complicated strategies. This simple strategy is called “tit-for-tat.” To use this strategy, start with C, and then for each subsequent turn, copy the opponent’s choice from the previous move. The player “rewards” the opponent who cooperates by cooperating on the next move and “punishes” the opponent who defects by defecting on the next move. Nobody has ever proved that this is the best strategy, but statistically, it has dominated competitions.

3. Play another round with Player A using the same strategy as the first time, but this time Player B should use “tit-for-tat.” Describe the results of this game.

4. Play a third round with A and B switching roles from Question 3. Who wins this time?

5. What is the result if both players use the “tit-for-tat” strategy?

6. What is the result if both players use “tit-for-tat,” but on one move somewhere in the game, player A defects when the strategy calls for cooperation?

7. If both players begin with “tit-for-tat,” what strategy should a player change to in order to guarantee a victory?
Darwin Plays Game Theory—and Wins

05.22.2009

A computer simulation predicts that ravens should have evolved a behavior called "gang foraging," which is then observed in real ravens.

by Andrew Grant

Game theory, the branch of mathematics best known for exploring economics, has for the first time successfully predicted animal behavior in nature. It forecast a foraging strategy for ravens that was later observed in the wild.

Game theory analyzes how people (or animals) act in situations where an individual's success depends on both his own decisions and those of others. In 2002 Sasha Dall, a mathematical ecologist at the University of Exeter in England, used a game theory model to explain why young ravens scout for carrion alone but then recruit other birds to join the feast. This apparently altruistic behavior is evolutionarily sensible, he found, because it helps the scout fight off territorial adults and secure dominance over recruits.

Dall’s model predicted another successful strategy, one that had never been observed in ravens: gang foraging, in which a large group of birds scavenge together. Within a year, behavioral ecologist Jonathan Wright of the Norwegian University of Science and Technology discovered this very behavior in the field. He tracked ravens in North Wales by implanting carcasses with different-colored beads that the birds ingested and later coughed up. Analysis of the beads indicated that ravens in some roosts were searching, eating, and benefitting together, just as Dall anticipated. Wright and Dall merged their mathematical and field-based work in a paper published in February.

Charles Darwin noticed that animal teamwork could make evolutionary sense for groups, but Wright's research shows that joint efforts can pay off even in situations that emphasize the role of individual success. “Ravens are smart and selfish,” Wright says, “but there’s a lot more stability to cooperation.”
Given the odds, ethics, would you go for the extra credit?

By Barry Balof
Assistant Professor of Mathematics

When faced with an election, are we better off choosing a candidate who will raise taxes to implement more governmental programs? Will our individual votes match what we feel is best for the community? On an international scale, what roles do calculation and psychology play in our economic and military decisions with regard to other nations? How should an overcommitted student, as so many of them are these days, decide to allocate his or her precious time?

Quandaries like these require a measure of mathematical ability as well as psychological awareness. During my fourth year of graduate school, I left the confines of my mathematical immersion to take a course in the psychology department on decision-making. During this course, we spent the first day and a half discussing the optimal way to make a decision and the rest of the term discussing why people don’t do it that way. Mathematics plays a large role in the “optimal way,” but we often find that mathematical considerations are trumped by our psychological tendencies. I’ve incorporated some decision-making exercises into my classes both to help me gain insight into how my students think and to show them something about how psychology affects their own thought processes and mathematical decisions in areas beyond my classroom.

Below are three examples of these exercises. I’ll often give such a question at the end of a quiz. We’ll typically analyze it in the next class, discussing the optimal solution, the outcome chosen by the class and the (many) differences between the two.

What Would My Neighbor Do?

Extra credit: Would you like one point or three points of extra credit? Note: If more than 25 percent of you choose three points, no one gets anything. Students have just toiled through calculations of derivatives, integrals or vectors and are now faced with the chance to improve their grade. A response of one point increases the likelihood that the whole group will get something, but as individuals, they might be able to afford to go for the three points, trusting their peers to be more benevolent (or conservative) in their decisions.

So, what would you do? How do you think they did?

In general, my classes will not get any extra credit, and, in fact, they will miss it by a mile. In the last class that had this question, 50 percent of the students opted for three points. One student gave the following rationale for his decision: “I know that we should choose only one point for the benefit of the group, and that my grade could probably afford it (it could).” I would rather that no one get anything than that other students get more than I do. Not too many in the class were that convinced, however, the scores will plummet.

What should you do? What would you do? How did the class do?

Again, approximately 50 percent of the group opted for “greed,” earning three points, while the other half opted for benevolence, earning two points. Strikingly, there was no correlation between student choices and such factors as year in school (the group was mostly first years, still new to college, and sophomores, who may be more attuned to the way their peers think), gender or even standing in the course. (Those who were doing well in the semester were equally likely to choose for themselves or choose for the group). Again, students commonly and prematurely began to worry about what others would do for themselves and not have to watch others do better than they did.

Half Full?

Lest you think that all Whitman students are so cynical when it comes to matters of how their peers think, I present the following example based on a short-lived game show, “Friend or Foe” (which itself was based on the classic strategy game Prisoners’ Dilemma):

Extra credit: Complete the following quiz with a partner. The score that the two of you earn is extra credit, but you must decide individually how much of the credit you’d like. You can opt for either “half” or “all” of the credit. If you both choose “half,” then you’ll split the points. If one of you chooses “half” while the other chooses “all,” then the person who chose “all” gets all the credit, while the other gets nothing. If both of you choose “all,” then neither of you gets any credit.

This example continues the theme of the difference between individual and group strategy, but the fact that the group size is now only two strengthens the psychological component. Individually, each person is better off trying for the whole amount of extra credit, but if both partners follow this individual strategy, both walk away empty-handed. For this exercise, students were paired randomly, so that the “friend” incentive to cooperate was lessened. As context, the game show was able to do, the students were given a brief time after the quiz to talk to their partners and convince them to “do the right thing.” It is easy to see how a carefully constructed mathematical argument in this scenario might give way to a “gut feeling” about one’s partner.

How did the class fare?

The arguments must have been convincing, because the students universally opted to split the credit with their partners. Perhaps the earlier exercises had opened the group to the consideration of the whole over themselves, or perhaps, students find it harder to act selfishly when dealing with one individual rather than the nicholas “group.” What is striking is that these data differ from other groups performing similar tasks. In previous studies, participants opted nearly half of the time to act selfishly, which is also in accordance with the amount of money given out by the game show during its short run. As a result, I don’t necessarily expect the cooperation to be quite so pronounced in future groups, but I am heartened by the results thus far.

While these exercises are self-contained and may seem largely mathematical on a first reading, their relevance to other areas is quite clear. The examples on tax policy, foreign relations and time management have both mathematical and psychological considerations, and it’s important to understand how the two are interrelated.

So much of a liberal arts education revolves around cross-disciplinary thinking. Through these exercises, students see a different applicability of mathematics — and if they trust their peers — earn a little extra credit as well.

Barry Balof joined the Whitman faculty in 2003. He received his master’s and Ph.D. degrees from Dartmouth College.
FORGIVENESS MATH

The following pages contain an article from Discovery magazine (from 1993). Make copies to distribute to students for reading at home. The final two pages are questions for the students to answer based on the reading and to turn in for homework.

ANSWERS

1. In this article the author says that nice guys will win out but they need the help of not so nice guys along the way. Why did the need help? What happened to the not so nice guys?

   They need the help of the retaliators (not so nice guys) or police along the way to help with other not so nice guys, then forgiveness can start to take hold and the not so nice guys go away.

2. Does Darwinian theory account for the survival of species in which members show kindness and caring toward each other? Why or why not?

   No, Survival of the fittest says keep as much food and resources for yourself and family group.

3. One species that does take care of its own is the vampire bat that shares food. What is the result of their sharing? Would a human family unit have a similar structure and outcome? Justify your answer.

   The result is that the mortality rate drops from 84% to 24%. A human family probably already does this, but extending t to larger groups or tribes or to fellow countrymen would seem to have the same effect.

4. In this article the Prisoners Dilemma gives out point values of 0, 1, 3, 5. How would you describe the difference between a 1 and a 3?

   The 3 is when both cooperate and the 1 value is what you get when both defect.
5. Mathematicians play this game on a computer over and over. Consequently they can change strategies and view outcomes quicker. What should you do if you play against someone who always cooperates? 

*Always defect and you gain 5 points every time.*

6. After the first running of prisoners’ dilemma where 14 different strategies were used, what strategy worked best? What did this strategy do? 

*The Tit for Tat strategy worked best. This strategy cooperated for the first round then mirrored everything the opponent did.*

7. When forgiveness was incorporated into this strategy what was it called and what was the outcome? 

*Tit for Two Tats. This strategy cooperated for two defections, but it encouraged more cooperation over the long run and had the best outcome. However in another running of strategies, people developed really mean strategies. They knew that Tit for Two Tats would only retaliated after two defects, so the new strategy was to defect every other round and then Tit for Two Tats lost big.*

8. In 1980 Axelrod changed the game once again to try to mimic natural selection better. What did he do to change the game? Who won this round? 

*He changed the game so that the winner would reproduce with offspring who would follow the same strategy. Losing strategies would die out. Tit for Tat won in the long run.*

9. Tit for Tat did well when it started first by combining in clusters or families. Why did this help? 

*Because this allowed for reproduction of offspring with the same strategy. Mean strategies like Always defect would only be rewarded with 1 offspring at a time and eventually lost.*

10. What is a generous Tit for Tat strategy?
It followed pretty much the Tit for Tat but it allowed for a little flexibility and occasionally forgave a defection.

11. What are stochastic strategies and what happened the first time this stochastic simulation was run with generous Tit for Tat?

Stochastic strategies have a degree of randomness in them that allow for flexibility and they don’t necessarily follow the rules exactly. The running in 1980 had a random generator involved, it chose who was playing whom at any given time. Tit for Tat started taking over and the mean strategies died out.

12. Mathematicians felt there was a flaw in the above running. What did they figure went wrong and how did they change the flaw? What was the outcome?

At first they realized that the random generator was flawed and so it wasn’t allowing all strategies equal playing time. When they fixed that, the game went crazy and it deteriorated into Always defect. Nature didn’t work like that. When they allowed for a small batch of Tit for Taters to group at the beginning then they won out and Always Defect lost, but they needed a small head start. They needed some retaliators to help them start, then they took over, but even then they eventually lost to Generous Tit for Tat.

Evolution, in our dog-eat-dog world, should have made short work of unselfish behavior. Yet such behavior exists, and mathematical games suggest how it got its start.

by Thomas A. Bass

Nice guys do not always finish last. In fact, they sometimes finish first. And now we have scientific evidence to prove it, thanks to the work of two Austrian mathematicians who have discovered the value of forgiveness. Or at least they’ve discovered how forgiveness might have come into being in our dog-eat-dog world.

Generosity pays off under conditions of uncertainty. You should not be too tolerant, but not too intolerant, either, says Karl Sigmund, a 47-year-old mathematician at the University of Vienna. Never forget a good turn, but try occasionally to forgive a bad one. We benefit from cultivating a keen sense of gratitude dosed with a small amount of generosity.

The American embassy, bristling with listening posts, lies outside Sigmund’s office window. Just over the border, warring Serbs and Croats are killing each other to the south, while the former Soviet empire is imploding to the east. No wonder the evolution of cooperation is a hot subject in Vienna, where this past year Sigmund and his former graduate student Martin Nowak codiscovered what one might call the Viennese Golden Mean.

In spite of its happy ending, this story about how nice guys came into existence and survived is fraught with near misses and harrowing escapes. And it turns out that nice guys can’t do it on their own. They need help along the way from some not-so-nice guys, who will then disappear in a final apocalyptic outbreak of good feeling.

I by no means want to give the impression that this tendency toward generous cooperation is the usual thing, says Sigmund, who has wild shocks of hair standing up on top of his head, a bottle-brush mustache, and spectacles. As he maps out the limits of this new theory of forgiveness, he sits under an etching of Captain Nemo steering a submarine 20,000 leagues under the sea. It only works after the cooperators get help from stern retaliators. This is the basic message: To get cooperation you need a police force, but then the police die out. So it’s good to have police but not to be police! If all this sounds strange and confusing, welcome to the bizarre world of the mathematics of forgiveness.

Scientists have long been mystified as to why anyone would ever do something unselfish for someone else. These displays of niceness are called altruism, and they don’t seem to square with the Darwinian scheme of things. The point of the evolutionary game is to pass along your
genes, and your best chance to do that usually means grabbing as much food and other resources for yourself and your progeny as you can. Animals with unselfish, generous impulses would seem ill-equipped to compete and likely candidates for a quick death. Darwin called it the survival of the fittest—not the nicest.

So why do vampire bats share their blood meals with unrelated, less fortunate neighbors? A vampire bat has to consume between 50 and 100 percent of its body weight in blood every night. It will die if it fails to feed for two nights in a row. But a bat on the edge of starvation can gain 12 hours of life and another chance to feed if it is given a regurgitated blood meal by a roost mate. Someone has actually worked out the odds on this vampire buddy system. If the bats didn’t practice food sharing, their annual mortality rate would be 82 percent. With food sharing, this rate drops to 24 percent.

Bats aren’t alone in their generosity. Food sharing among unrelated individuals is also practiced by wild chimpanzees and occasionally by human beings. Stickleback fish team up to inspect dangerous predators. Black hamlet fish, which have both male and female sex organs, take turns fertilizing each other’s eggs; one fish, after being fertilized, could selfishly cut and run, leaving its former partner in the lurch.

After the Second World War, the biological debate about why altruism exists got picked up by social scientists. If vampire bats could get their act together, wasn’t there hope for the Kremlin and the Pentagon? The most famous paradigm for modeling the evolution of cooperation among selfish individuals is a mathematical game called the prisoner’s dilemma. It works like this:

Imagine two prisoners facing life in the slammer for a crime they committed together. They are questioned separately by the authorities. If they resist the temptation to rat on each other, their alibi will hold and they’ll both be released after a few months in jail. (Let’s assign this outcome a value of 3 points each, with the object of the game being to score the most points.) If both prisoners chicken out and rat on each other, they’ll each get a longer sentence (albeit less than the maximum because they get time off for turning state’s witness); this lower payoff is worth 1 point each. But the highest payoff goes to the prisoner who rats while his buddy remains silent; then the ratter goes scot-free, for 5 points, while the silent sucker gets the maximum sentence, for 0 points.

In the simple prisoner’s dilemma, where the game is played only once and no future matches are envisioned, the rational choice about how to act is so clear that there is really no dilemma at all. If you’re nice, you risk your opponent’s playing you for a sucker; the only way to minimize the risk is to rat. You’ll get at least 1 point, and as many as 5 points if your opponent is a gullible fool. Sorry, nice guys. No grounds for hope yet.

But this model of life, which pictures the world as a dark alley full of strangers meeting only once in their lives, is not particularly realistic. It is far more likely that we keep meeting the same boyz in the hood. Mathematicians have taken account of this fact by iterating, or
repeating, the prisoner’s dilemma, so that the same players face each other again and again.

When you play repeated rounds of the prisoner’s dilemma, the game is completely different. The most surprising fact is that the competition no longer has a single strategy that is better than all the others. The game becomes contingent, chancy. Everything depends on whom you are playing at any particular moment.

For the simple prisoner’s dilemma you have one best strategy, which is to defect, says Sigmund. But for the repeated prisoner’s dilemma there is no best strategy, and it is impossible that a single one could ever be found. You can always encounter situations where it’s better to switch strategies. It all depends on what kind of player you’re partnered with.

Consider the following example. If you meet a relentless defector, you should always defect. If you meet an all-out cooperator, you should also always defect. But if you meet a grim retaliator—someone who cooperates until his opponent defects and from that moment on never cooperates again—you should cooperate.

Two mad-dog defectors hammering each other without surcease will pick up only 1 point per round, while two cooperators, consistently scratching each other’s backs, will rack up a steady 3 points per round. We begin to see how cooperation can pay off, and in the long run, with the arbitrary values we have assigned to the game, 3 points is the highest average payoff anyone can expect to earn per round.

Mathematicians, game theorists, biologists, and arms negotiators had been battling one another for 30 years, arguing about which strategy was best for playing repeated rounds of the prisoner’s dilemma, when Robert Axelrod, a political scientist at the University of Michigan, decided to settle the matter with a computer tournament. Unlike humans, who get bored and sloppy when working at this level of detail, computers can play these strategies against each other ad nauseam, or at least until the plug gets pulled.

Researchers around the world mailed Axelrod 14 different computer programs. He added one of his own and played all of them against each other in a round-robin tournament in 1978. The winner was the simplest program, Tit for Tat. It came from Anatol Rapoport, a former concert pianist and one of the grand old men of game theory. Rapoport had political reasons for being interested in the prisoner’s dilemma, says Sigmund, who met him when Rapoport moved from Canada to Vienna in the early eighties. Rapoport was strongly committed to the peace movement, which may be one of the reasons he came to a neutral country like Austria.

Rapoport saw the arms race between the superpowers as one of the most spectacular examples of the prisoner’s dilemma. And a logical arms-race strategy, Rapoport reasoned, was Tit for Tat: cooperate in the first round and then imitate whatever the other player does.

As successful as it may be in some situations, Tit for Tat is not always the best strategy. On analyzing the results of his first tournament, Axelrod found that Tit for Tat could have been
beaten by another strategy that no one had thought to enter: Tit for Two Tats. This more generous policy calls for retaliation only after two consecutive rounds of defection from the other player. Its forgiving nature elicited more cooperation from other players over the long run, so that even though Tit for Two Tats wouldn’t win any single encounter, overall it would rack up the most points. In contrast, stricter strategies such as Tit for Tat would be forced into battling retaliators, thereby getting a low number of points and falling behind Tit for Two Tats in scoring.

Axelrod held a second round-robin tournament in 1979. Tit for Two Tats was one of 63 programs entered. Although it would have won the first tournament, this time the strategy was manifestly trounced, finishing twenty-fourth. What happened was that people entered nastier strategies, which were designed to take advantage of nice guys. They knew Tit for Two Tats would retaliate only after two defections in a row, so these mean strategies simply defected every other round. They racked up a lot of points, and Tit for Two Tats could neither retaliate nor tempt the others to cooperate. This shows how contingent the game can be, the outcome changing with every unique constellation of players.

So who was the winner? Trusty old Tit for Tat, which once again showed how well it does against selfish players. If a player cooperates, so does Tit for Tat; if the player defects, Tit for Tat does, too. There’s little room to take advantage of it. The selfish player can’t get too far ahead, and Tit for Tat grabs a lot of points when playing other cooperators. Overall, it gets the highest point total.

In 1980 Axelrod came up with a different kind of competition. He wanted to use computers to simulate natural selection by modeling ecological encounters in nature. Participants in this ecological tournament formed a population that altered with each repetition of the game. The strategies that got the most points during Round One, for instance, would be rewarded with offspring: two or three other versions of themselves, all of which would participate in the next round. In this way one can build up entire populations of strategic cooperators or defectors. During successive rounds, winning strategies multiplied while less successful rivals died out.

This is the tournament that first got Karl Sigmund interested in the prisoner’s dilemma. Sigmund had been working on theoretical chemistry at the University of Vienna, studying the hypercycle, a system of self-reproducing molecules that might hold clues to how life evolved on Earth. I got very excited when I found the prisoner’s dilemma had meaning in evolutionary biology, says Sigmund. He got so excited, in fact, that he switched from studying self-replicating molecules to studying models of animal behavior.

For a mathematician, says Sigmund, whether you study molecules or the behavior of animals doesn’t matter. It all reduces to the same differential equations. Mathematically this is really one field: the population dynamics of self-replicating entities. They can be RNA molecules or reproductive strategies or animals preying on each other or parasites or whatever. Success determines the composition of the field, and the composition determines success. It is not easy to predict where this may lead.
Sigmund watched with interest as Axelrod played his ecological tournament out to the thousandth round. Again, Tit for Tat was the winner. At first glance this may appear to be paradoxical, but of course it isn’t, says Sigmund. When thrown into a hornet’s nest of inveterate defectors, a single Tit for Tat will do less well than the meanies, because it loses out on the first round before switching into tough-guy mode. But when playing itself or other nice strategies, Tit for Tat will do significantly better than such hard-liners as Always Defect, which can’t get more than one point per interaction with itself. The moral is that Tit for Tatters do best when they start interacting in clusters or families, Sigmund says. Kinship facilitates cooperation. In a mixture of Always Defect and Tit for Tat, even if only a small percentage of the population is using the nice policy, that policy will start reproducing itself and quickly take over the game.

Even as Tit for Tat was racking up one success after another, it was clear to Axelrod and Sigmund that the strategy had a fatal flaw. It has no tolerance for errors, says Sigmund. While computer programs interact flawlessly, humans and other animals certainly do not. In biological or human interactions, it is clear that sometimes the wires get crossed and you make a mistake about the identity of someone. You meet a friend and, not recognizing him, you defect. This is the Achilles’ heel of Tit for Tat, which is particularly vulnerable against itself.

A realistic version of Tit for Tat against Tit for Tat can fall into endless cycles of retaliation. Since all it knows how to do is strike back at defectors, one scrambled signal will send Tit for Tat spiraling into vendettas that make the Hatfields and the McCoys look tame by comparison. The average payoff for Tit for Tat drops by 25 percent if you introduce only a few such mistakes into the tournament. This is a terrible performance, Sigmund says, noting that a random strategy (unthinkingly defecting or cooperating in each round with equal probability) will do just as well.

The obvious way to break up this vicious cycle of grim retaliation consists in being ready, on occasion, to let bygones be bygones. We can even compute the optimal measure of forgiveness, says Sigmund. The Viennese Golden Mean—the extra dose of generosity that makes for the best of all strategies in a less-than-perfect world—is contained in the following rule: Always meet cooperation with cooperation, and when facing defection, cooperate on average one out of every three times. The strategy that embodies this rule is called Generous Tit for Tat.

The merits of Generous Tit for Tat were already known in the early 1980s, when a Swedish scientist named Per Molander computed the benefits of generosity in a world where pure Tit for Tat could sometimes make a mistake—like Oedipus meeting his father. Molander arrived at the figure of one-third through repeated rounds of trial and error. Molander’s finding does not mean that you should turn the other cheek to every third blow. Obviously it would be a big mistake to let your opponent know exactly when you were going to be nice. The number is just an average.
Generous Tit for Tat looked great on paper, but Sigmund wanted to know if it bore any resemblance to reality. Was there any proof that an evolving population--be it molecules, animals, or strings of numbers in a computer program--would actually adopt Generous Tit for Tat? We wanted to know whether Generous Tit for Tat was biologically relevant, Sigmund says. You could easily show it was the best strategy for a group, but would evolution necessarily lead toward this best strategy?

To answer the question, Sigmund and Martin Nowak organized a computer tournament of their own in 1991. Nowak had been a chemistry student at the University of Vienna when he heard Sigmund give a lecture on the prisoner’s dilemma. He found the problem so intriguing that he allowed himself to be kidnapped by Sigmund and converted into writing a dissertation on the mathematics of the game. Nowak finished the work in a year--a university record--before heading to Oxford, the mother church of theoretical biology.

Realizing how important mistakes can be to the prisoner’s dilemma, Sigmund and Nowak started playing the game with what are called stochastic strategies. Stochastic means random, and strategies such as these--Generous Tit for Tat being one of them--allow for reacting to one’s opponent with a degree of flexibility.

We had been experimenting with games where everyone was playing the same strategy and watching what would happen when a small minority came in and started playing something else, says Sigmund. Would they spread or be wiped out? Later we got the idea, why not try out our model with 100 different strategies, chosen at random to be more or less tolerant, more or less forgiving? Some would forgive one out of two times, some one out of five, and so on. And some, of course, would never forgive.

All Sigmund and Nowak needed to organize their tournament was a random number generator, something to tell the computer which two strategies to face off against each other at a given moment. But we didn’t have a random number generator, Nowak says. So I just made one. Nowak came to Vienna over Christmas, and the two men loaded their players, including Generous Tit for Tat, into Nowak’s laptop computer. Then they sat back and watched the players duke it out for thousands of generations. Winning strategies reproduced themselves. Losers got knocked out of the ring.

Everything was looking very bright, going just the way we expected, says Sigmund. The more generous strategies invariably gained the upper hand. But when Nowak got back to Oxford and started working on a proper computer, we discovered we had been using the wrong random number generator, so we weren’t really representing every possible percentage of retaliation. Nowak explains: The one I made was biased. Some strategies came up more often than they should have. Tit for Tat was one of them.

When Nowak fixed the problem, giving all the strategies a level playing field, things got out of hand. Instead of nice strategies taking over the game, evolution consistently veered to Always Defect, which we didn’t like at all, Sigmund says. I mean, nature doesn’t work that way. It
wasn’t realistic.

After two black weeks of watching meanies invade their tournament, Nowak and Sigmund accidentally stumbled on the key to evolving cooperation. They noticed the game took a radically different turn with one small alteration. If a dose of Tit for Tat—just enough to establish a tiny enclave, a dose as small as one percent of the population—was added to the game at the start, then it flipped directions. Generous Tit for Tat is not strong enough to organize this emergence of cooperation, says Sigmund. What one needs is a kind of police force, a minority that helps by its very strictness to effect this move but that does not ultimately prove the best.

If you don’t have Tit for Tat but some other strategy, it cannot do it. It must be a very strict retaliator. But then after you have the switch toward cooperation, it is not Tit for Tat that profits. Its frequency goes up, but then it yields to Generous Tit for Tat. Tit for Tat is not the aim of evolution, but it makes it possible. It is a kind of pivot.

For 100 generations the Always Defect strategy dominates the population with what looks like inescapable ferocity; it looks so bad you almost give up hope. A beleaguered minority of Tit for Tat survives on the edge of extinction. But when the suckers are nearly wiped out and the exploiters have no one left to exploit, the game reverses direction. The retaliators spring back to life. The exploiters suffer crippling reverses. There was great pleasure, Sigmund recalls, in watching the Always Defectors weaken and then die out.

But the staunch Tit for Tatters are not the ones who ultimately win the game. They will lose out to their even nicer cousins, who exploit Tit for Tat’s fatal flaw of not being forgiving enough to stomach occasional errors. After 100 generations the game swings from nasty to nice, and after 300 generations it swings again to extra-nice, with Generous Tit for Tat so firmly established that no meanies can invade the game.

When every player is employing Generous Tit for Tat, reproducing itself without a worry in the world, the game has arrived at the end of history. Evolution has effectively stopped for these self-perpetuating nice guys.

This is a pleasant image, but Sigmund doesn’t believe such states last for long. Only in a limited strategy space do you reach the end of history. But the space can always be enlarged by adding strategies with more memory or other features. Organisms can build up experience by watching each other and sharing information among themselves. Evolution is certainly not going to stop. The number of possible strategies in this game is fantastic.

Nor is it a complete picture of the biological world. This is a very simplified model of confrontation, he says. Out of all the imaginable interactions in the world, very few reduce to the prisoner’s dilemma. It is not the most universal, or even the most common, design of interactions found in nature. But it’s so simple. It’s transparent. It’s like a haiku. The world is reduced to a couple of lines.
Yet it is satisfying to know this is one possible path by which cooperation and selflessness could have been established on this planet. So after publishing their findings in Nature last year, Sigmund and Nowak are already planning more tournaments. The problem that most interests me now is to get longer memory into play, says Sigmund. We would like to model how you can build up trust by repeated interactions, and this building up of trust is something that can happen only when one has longer memory. I think the strongest tendency in natural selection is toward evolving a better memory, but the message of Generous Tit for Tat is that it may also pay to forget.
1. In this article the author says that nice guys will win out but they need the help of not so nice guys along the way. Why did the need help? What happened to the not so nice guys?

2. Does Darwinian theory account for the survival of species in which members show kindness and caring toward each other? Why or why not?

3. One species that does take care of its own is the vampire bat that shares food. What is the result of their sharing? Would a human family unit have a similar structure and outcome? Justify your answer.

4. In this article the Prisoners Dilemma gives out point values of 0, 1, 3, 5. How would you describe the difference between a 1 and a 3?

5. Mathematicians play this game on a computer over and over. Consequently they can change strategies and view outcomes more quickly. What should you do if you play against someone who always cooperates?

6. After the first running of prisoners’ dilemma where 14 different strategies were used, what strategy worked best? What did this strategy do?
7. When forgiveness was incorporated into this strategy what was it called and what was the outcome?

8. In 1980 Axelrod changed the game once again to try to mimic natural selection better. What did he do to change the game? Who won this round?

9. Tit for Tat did well when it started first by combining in clusters or families. Why did this help?

10. What is a generous Tit for Tat strategy?

11. What are stochastic strategies and what happened the first time this stochastic simulation was run with generous Tit for Tat?

12. Mathematicians felt there was a flaw in the above running. What did they figure went wrong and how did they change the flaw? What was the outcome?

13. Did your highlights and notes help you answer these questions? Why or why not?

14. In the article from Discovery, what similarities were there in the findings about ravens and the findings in the Forgiveness Math article?

15. What is the value in reading articles like this when dealing with real life conflicts? Justify your answer.